A Neuro-Inspired Method for Data Rate Limited Feedback Control

Andrew Lamperski

Abstract—The nervous system implements a networked control system in which the plants take the form of limbs, the controller is the brain, and neurons form the communication channels. Unlike standard networked control architectures, there is no periodic sampling, and the fundamental units of communication contain little numerical information. This paper describes a novel communication channel, modeled after spiking neurons, in which the transmitter integrates an input signal and sends out a spike when the integral reaches a threshold value. The receiver then filters the sequence of spikes to approximately reconstruct the input signal. It is shown that for appropriate choices of channel parameters, stable feedback control over these spiking channels is possible. Furthermore, good tracking performance can be achieved. The data rate of the channel increases linearly with the size of the inputs. Thus, when placed in a feedback loop, small loop gains imply a low data rate.

I. INTRODUCTION

Nature hints at solutions to many of the problems that engineers face, and control systems are no exception. Take, for example, the act of reaching for an object. Sensors in the eyes and muscles take measurements which are passed to the brain by short lived voltage spikes, called action potentials, that travel through neurons [1]. Using this information, the brain then sends a sequence of spikes through effector neurons to the muscles which generate the desired motion.

For the past decade, networked and data rate limited control systems have been fruitful areas of research [2], [3]. The typical setup of these systems involves controlling a plant which is located remotely from either the sensors or the controller. In such systems data going into and/or out of the plant must be sent through a communication network (possibly with limited data rate). In the reaching example above, the controller is the brain, the sensors are the eyes and proprioceptors in the muscles [4], the plant is the arm, and neurons passing between the brain and the arm serve as communication channels.

The networked control scheme implemented by the nervous system differs drastically from the methods typically employed in the control systems literature. First, there is no periodic sampling. A neuron sends a spike signal when the voltage across the cell membrane reaches a certain threshold. Thus the control scheme is event-triggered, as in [5]. Secondly, the form of an action potential carries no information. All the information from a single neuron is encoded in the spike timing [1]. Additional information can, however, be gained by using multiple neurons and distinguishing between which neurons are sending spikes. Traditional networked control methods require that the information transmitted across the communication network be packets of (possibly quantized) numerical data. Thus, unlike existing methods for networked control, the basic unit of communication in neural control (the spike) conveys minimal numerical information.

This paper studies continuous-time networked control in which signals to and from the plant pass through a communication channel that was inspired by the behavior of spiking neurons. It does not attempt to describe how the nervous system implements control tasks.

The communication channel, termed the spike channel, was designed based on methods for reconstructing the input current to neurons by applying a linear filter to their spike sequences [6], [7]. The spike channel (Figure 1) works as follows. A continuous-time input is sent through a low-pass filter. When the state of the filter reaches a threshold value (either positive or negative) the state of the filter is set to zero, and a delta function of a fixed magnitude and appropriate sign is sent to an identical low-pass filter. The output of the channel is the output of the second filter. Surprisingly, even with this strong spiking nonlinearity, the spike channel behaves like a low-pass filter, up to a bounded additive disturbance (Figures 4 and 5).

The main results of this paper describe stability, tracking performance, and data rate for feedback control of a continuous-time LTI SISO plant by a continuous-time LTI SISO controller when signals to and from the plant pass through the spike channel (Figures 2 and 3). In particular, if the nominal feedback loop is internally stable, then spike channel parameters can be chosen so that internal stability is preserved. By appropriate choice of parameters, tracking performance in the spike channel system can be made arbitrarily close to that of the nominal system. The price of good tracking and stability properties manifests in higher spike rates.

The control system analysis techniques used in this paper are standard and can be found in textbooks such as [8], [9].

II. PRELIMINARIES

This section defines the notation used throughout the paper, describes the problem of interest, gives a neurobiological
The imaginary number \( RC \)-circuit equations: integrate-and-fire model, is given by the following modified C. Neurobiological Motivation

motivation for the communication channel introduced, and then defines the communication channel.

A. Notation

Most of the notation is standard. The real numbers are denoted by \( \mathbb{R} \). For a function, \( x : \mathbb{R} \to \mathbb{R} \), the \( L_1 \) and \( L_\infty \) norms are denoted by \( \|x\|_1 \) and \( \|x\|_\infty \) respectively. Let \( x(t^-) = \lim_{t \to t^-} x(s) \) and \( x(t^+) = \lim_{t \to t^+} x(s) \) if the limits exist. The imaginary number \( j = \sqrt{-1} \).

B. Problem Formulation

This paper studies a networked control system in which communications to and from the plant occur via the spike channel. Consider the feedback loop depicted in Figure 2. As is standard in classical control, \( P \) is a strictly proper continuous-time SISO transfer function, and \( C \) is a proper continuous time SISO transfer function. As is common in networked control, assume that the plant is separated from the controller, and thus inputs to the plant and measurements from the plant must pass through communication channels. The particular communication channel used is the spike channel described later in this section (Figure 3).

The goal of this paper is to study the stability, tracking performance, and data rate of the feedback loop with spike channels.

C. Neurobiological Motivation

A common model for a single neuron, known as the leaky integrate-and-fire model, is given by the following modified RC-circuit equations:

\[
\begin{align*}
\dot{V}(t) &= -\frac{1}{RC} V(t) + \frac{1}{C} I(t) & \text{if } V(t) < V_{th} \\
V(t^-) &= 0 & \text{if } V(t^-) = V_{th} \\
V_{out}(t) &= \sum_{\{t \leq t : V(t^-) = V_{th}\}} \delta(t - \bar{t}).
\end{align*}
\]

The state variable \( V \) corresponds to the electrical potential across the cell membrane of the neuron, and \( I \) is an input current. Thus, the neuron integrates the potential with a leak term proportional to \(-\frac{1}{RC} V\). When the potential reaches some threshold, the electrical potential across the membrane is set to zero and a spike (approximated as a delta function) is sent as an output.

Thus, the leaky integrate-and-fire model can be viewed as an input-output mapping \( H_{\text{LIF}} \) that takes an input current \( I \) and outputs a sequence of spikes \( V_{out} \).

Note that the operator \( H_{\text{LIF}} \) only generates outputs for positive currents. A simple method for obtaining information about both positive and negative input current is to examine the output of a pair of opposing leaky integrate-and-fire neurons, \( H_{\text{LIF}}(I) - H_{\text{LIF}}(-I) \). It has long been known that good approximations of the input current can be obtained by applying a well chosen linear filter to the spike sequence \( H_{\text{LIF}}(I) - H_{\text{LIF}}(-I) \) [6]. In other words, there is a linear filter \( G \) such that, for well behaved input signals, \( G(H_{\text{LIF}}(I) - H_{\text{LIF}}(-I)) \approx I \).

The work in this paper is motivated by the idea that the signals resulting from an integrate-fire-reconstruct process, similar to that described above could be good enough for feedback control.

Remark 1: Many treatments of leaky integrate-and-fire neurons include an extra dynamical mode, called a refractory period. If a spike occurs at time \( t \), then \( V(t^+) = 0 \). In the model above, immediately after time \( t \), the membrane begins to integrate current. If a refractory period is modeled, then for a time interval \((-\tau_{ref}, t] \), \( V \) is held constant at 0. While including a refractory period makes the neuron model more biologically realistic, this paper neglects refractory periods in the interest of analytical simplicity. See [1] [7] for more on neuron models with refractory periods.

D. The Spike Channel

This paper proposes a new communication channel, termed the spike channel, that mimics the integrate-fire-reconstruct process observed to be successful in approximating input currents to spiking neurons.

The spike channel is an input-output mapping \( H \) taking input \( w(t) \) to output \( z(t) \) based on the following rules:

\[
\begin{align*}
\dot{x}_1(t) &= -\frac{1}{\tau} x_1(t) + \frac{1}{\tau} w(t) & \text{if } |x_1(t)| < r \\
\dot{x}_2(t) &= -\frac{1}{\tau} x_2(t) & \text{if } |x_1(t)| < r \\
x_1(t^+) &= 0 & \text{if } |x_1(t^-)| = r \\
x_2(t^+) &= x_2(t^-) + x_1(t^-) \\
z(t) &= x_2(t).
\end{align*}
\]

So, the spike channel consists of two first order low-pass filters with equal time constant \( \tau \), such that when the magnitude of the state of the filter reaches the threshold, \( r \), it is immediately set to zero and a delta function (or “spike”) of magnitude \( r \tau \) is applied to the second filter (Figure 1).

To understand the spike behavior more explicitly, assume that a spike occurs at time \( t \). Then the second line of equation (2) can be viewed as the application of a delta function as follows:

\[
x_2(t^+) = x_2(t^-) + \frac{1}{\tau} \int_{-\infty}^{t} e^{-(\sigma - t)/\tau} x_1(\sigma) \delta(\sigma - t) d\sigma.
\]
τ parameters set to Fig. 5. Response of the spike channel to input from Figure 4 with parameters were set to Fig. 4. Response of the spike channel to a randomly generated input. The x line and the output is the solid line. Bottom. The signal behaves like a low-pass filter, up to a bounded additive disturbance.

In the spike channel, \( w \) plays the role of the input current, and \( z \) can be thought of as the approximately reconstructed input. The special form of the spike channel leads to a straightforward quantitative analysis showing that \( H \) behaves like a low-pass filter, up to a bounded additive disturbance (Figures 4 and 5).

### III. Results

This section presents some key lemmas about the spike channel and uses them to derive the main results about feedback control using spike channels.

#### A. Spike Channel Lemmas

This subsection presents two preliminary results about the spike channel based on feedback schemes. The first lemma gives a bound on the spike rate (the number of spikes per unit time) based on \( r, \tau \), and the size of the input to the channel. The second lemma (Lemma 2), the most important preliminary result for this paper, shows that the output of the spike channel differs from the output of a low-pass filter by at most \( r \).

**Lemma 1:** If \( \|w\|_\infty = m \), then the spike rate is bounded above by

\[
f(m) = \begin{cases} \frac{1}{\tau} \ln \frac{m}{r} & \text{if } m > r \\ 0 & \text{if } m \leq r. \end{cases}
\]

Furthermore, \( f(m) \leq \frac{m}{\tau r} \) for all \( m \geq 0 \), and \( \lim_{m \to \infty} \frac{f(m)}{m} = \frac{1}{\tau r} \).

Lemma 1 is proved in the appendix.

**Lemma 2:** If \( w \) is bounded and \( y(t) = \frac{1}{\tau} \int_{-\infty}^{t} e^{-(t-\sigma)/\tau} w(\sigma) d\sigma \) is the output of a first-order low-pass filter with time constant \( \tau \), then

\[
y(t) = x_1(t) + x_2(t).
\]

In particular, \( |y(t) - z(t)| \leq r \), for all \( t \in \mathbb{R} \).

**Proof:** To find an expression for \( x_1(t) + x_2(t) \), it is useful to have expressions for each term, individually. Two cases arise: either infinitely many spikes have occurred up to time \( t \), or only finitely many have occurred. Only the infinite spike case will be proven since the finite case is similar. Let \( \ldots < t_{-2} < t_{-1} < t_0 \leq t \) be the times at which spikes occurred, up till time \( t \).

Since \( x_1 \) is reset to 0 at time \( t_0 \), \( x_1(t) \) is calculated to be

\[
x_1(t) = \frac{1}{\tau} \int_{t_0}^{t} e^{-(t-\sigma)/\tau} w(\sigma) d\sigma.
\]

On the other hand, from the spiking behavior defined in equation (2), the input to the second filter is

\[
\tau \sum_{k \leq 0} x_1(t_k^-) \delta(t - t_k).
\]

Therefore, the output of the second filter is

\[
x_2(t) = \frac{1}{\tau} \int_{-\infty}^{t} e^{-(t-\sigma)/\tau} \sum_{k \leq 0} x_1(t_k^-) \delta(\sigma - t_k) d\sigma = \sum_{k \leq 0} e^{-(t-t_k)/\tau} x_1(t_k^-).
\]

To make the expression for \( x_2(t) \) independent of \( x_1(t_k^-) \), the summand of equation (5) can be expanded as

\[
e^{-(t-t_k)/\tau} x_1(t_k^-) = e^{-(t-t_k)/\tau} \frac{1}{\tau} \int_{t_k-1}^{t_k} e^{-(t_k-\sigma)/\tau} w(\sigma) d\sigma = \frac{1}{\tau} \int_{t_k-1}^{t_k} e^{-(t-\sigma)/\tau} w(\sigma) d\sigma.
\]

Finally, combining equations (4-6) gives

\[
x_1(t) + x_2(t) = \frac{1}{\tau} \int_{t_0}^{t} e^{-(t-\sigma)/\tau} w(\sigma) d\sigma + \frac{1}{\tau} \sum_{k \leq 0} \int_{t_k-1}^{t_k} e^{-(t-\sigma)/\tau} w(\sigma) d\sigma = \frac{1}{\tau} \int_{-\infty}^{t} e^{-(t-\sigma)/\tau} w(\sigma) d\sigma.
\]
The second equality follows from the fact that $t_k - t_{k-1} \geq -\tau w_2(t) > 0$ (by Lemma 1), and thus $\lim_{t\to\infty} t_k = -\infty$.

Note that since spikes occur, it must be that $\|w\|_\infty > 0$.

**B. Main Results**

With the preliminary definitions set, the main results can be presented.

Lemma 2 implies that the spike channel can be conservatively approximated by a low-pass filter followed by an additive disturbance which is bounded by $r$. Thus by studying the feedback loop depicted in Figure 6, results about the stability, tracking, and data rate (because it is bounded by signal size) of feedback loop with spike channels can be inferred.

A feedback loop is said to be *internally stable* if whenever all the inputs (including disturbances) are bounded, then all the signals in the loop are bounded. The first result states that for small enough $\tau$, the spike channel preserves internal stability of the feedback loop.

**Theorem 1:** If the nominal feedback loop from Figure 2 is internally stable, then there exists $T > 0$ such that for all $\tau \in (0, T)$, the feedback loop with spike channels from Figure 3 is internally stable.

Theorem 1 follows immediately from Lemma 2 and the following lemma (which is proved in the appendix).

**Lemma 3:** If the nominal feedback loop from Figure 2 is internally stable, then there exists $T > 0$ such that the disturbance feedback loop from Figure 6 is internally stable for all $\tau$ such that $0 \leq \tau \leq T$.

The next result states that by choosing $r$ and $\tau$ small enough, the tracking error does not significantly degrade. Let $G_{e_1}^\tau$ be the impulse response of the mapping from input $u$ to tracking error $e$ for the system in Figure 6. Define $G_{e_1}^\tau$ and $G_{e_2}^\tau$ similarly. Let $G_{e_1}^\text{nom}$ be the nominal mapping from input $u$ to tracking error $e$.

**Theorem 2:** If $e_{\text{nom}}$ is the nominal tracking error, $e$ is the tracking error from the spike channel feedback loop and $\tau$ is small enough that the disturbance loop from Figure 6 is internally stable, then

$$
\|e_{\text{nom}} - e\|_\infty \\
\leq \|G_{e_1}^\tau - G_{e_1}^\text{nom}\|_1 \|u\|_\infty + (\|G_{e_2}^\tau\|_1 + \|G_{e_2}^\tau\|_1) r.
$$

Theorem 2 is an immediate consequence of Lemmas 2 and 3 combined with standard gain bounds.

Note that $\lim_{\tau \to 0} \|G_{e_1}^\tau - G_{e_1}^\text{nom}\|_1 = 0$. Thus, for small $\tau$, the difference in tracking performance is primarily due to the size of $r$, and for small $r$, the difference is due to the size of $\tau$.

Define $G_{w_1,u}^\tau, G_{w_1,d_1}^\tau, G_{w_2,d_2}^\tau, G_{w_1,u}^\tau, G_{w_2,d_1}^\tau$, and $G_{w_2,d_2}^\tau$ to be the input output mappings for the corresponding signals in Figure 6.

**Theorem 3:** If the nominal feedback is internally stable and $\tau$ is small enough that the disturbance loop from Figure 6 is internally stable, then the total number of spikes per unit time from both channels in Figure 3 is bounded above by

$$
\frac{\alpha(\tau)}{r} \|u\|_\infty + \frac{\beta(\tau)}{\tau},
$$

where

$$
\alpha(\tau) = \|G_{w_1,u}^\tau\|_1 + \|G_{w_2,u}^\tau\|_1
$$
$$
\beta(\tau) = \|G_{w_1,d_1}^\tau\|_1 + \|G_{w_1,d_2}^\tau\|_1
\|G_{w_2,d_1}^\tau\|_1 + \|G_{w_2,d_2}^\tau\|_1.
$$

**Proof:** By Lemma 1, the total spike rate is bounded by

$$
f(\|u_1\|_\infty) + f(\|w_2\|_\infty) \leq \|u_1\|_\infty + \|w_2\|_\infty.
$$

Now applying Lemmas 2 and 3, and $L_\infty$ gain bounds gives the result.

Thus if small $\tau$ and $r$ are chosen in order to maintain internal stability and good tracking, the data rate could become large.

Note that large values of $\alpha(0)$ or $\beta(0)$ correspond to large gains in the nominal system, whereas rapid growth of $\alpha$ or $\beta$ corresponds to fragility to the perturbation caused by inserting low-pass filters into the loop. If $\alpha$ or $\beta$ are large, then the data rates could be high, even in the case that $r$ and $\tau$ are large. Precise bounds on how $\alpha$ and $\beta$ vary with $\tau$ are beyond the scope of this paper.

On the positive side, if the loop gains from $d_1$, $d_2$, and $u$ to $w_1$ and $w_2$ are small, then the data rate is guaranteed to be small.

See Figure 7 for examples of the tracking response of the spike channel feedback loop. The unstable system requires a much higher spike rate (136.4 spikes per unit time) than the integrator system (21.3 spikes per unit time) to track the same input. Since the channel parameters, $r$ and $\tau$, as well as the input were identical, the difference in spike rate must be due to differences in loop gains.

**IV. CONCLUSION**

This paper describes feedback control over a novel communication channel, termed the spike channel. The spike channel is patterned after a configuration of neurons that allows simple reconstruction of the input current signal via
Thus the maximum spike rate is calculated to be

\[ f(m) = \begin{cases} \frac{1}{\tau \ln \frac{m}{m-r}} & \text{if } m > r \\ 0 & \text{if } m \leq r. \end{cases} \]

To calculate \( \lim_{m \to \infty} f(m) \), let \( \lambda = \frac{1}{m-r} \). Then since
\[ \lim_{\lambda \to 0} \frac{\ln(1+r\lambda)}{\lambda} = r, \]
the limit of the denominator of \( \frac{f(m)}{m} \) can be simplified as follows:

\[ \lim_{m \to \infty} m \ln \frac{m}{m-r} = \lim_{\lambda \to 0} \frac{1 + r\lambda}{\lambda} \ln(1 + r\lambda) \]

\[ = \lim_{\lambda \to 0} \frac{\ln(1 + r\lambda)}{\lambda} + \lim_{\lambda \to 0} r \ln(1 + r\lambda) = r. \]

Therefore \( \lim_{m \to \infty} \frac{f(m)}{m} \) is calculated to be

\[ \lim_{m \to \infty} \frac{1}{\tau m \ln \frac{m}{m-r}} = \frac{1}{\tau r}. \]

To see that \( f(m) \leq \frac{m}{\tau r} \) for all \( m > r \), note that

\[ f(m) \leq \frac{m}{\tau r} \iff m \ln \frac{m}{m-r} \geq r. \]

The calculations above show that \( \lim_{m \to \infty} m \ln \frac{m}{m-r} = r \), and furthermore \( \lim_{m \to r} m \ln \frac{m}{m-r} = \infty \). Thus, the proof can be completed by showing that \( m \ln \frac{m}{m-r} \) is monotonically decreasing.

\[ \frac{d}{dm} m \ln \frac{m}{m-r} = \ln \frac{m}{m-r} + m \left( \frac{1}{m-r} - \frac{1}{m-r} \right) \]

\[ = \ln \frac{m}{m-r} - \frac{r}{m-r}. \]

From the expression of the derivative, \( m \ln \frac{m}{m-r} \) is monotonically decreasing if and only if \((m-r)\ln \frac{m}{m-r} < r \) for all \( m > r \). Applying the change of variables \( \lambda = \frac{1}{m-r} \) gives

\[ (m-r) \ln \frac{m}{m-r} = \frac{\ln(1+r\lambda)}{\lambda} < r, \]

where the inequality follows from the first order necessary conditions for concavity. Thus \( f(m) \leq \frac{m}{\tau r} \) for all \( m \geq 0 \) and the proof is complete.

Proof: [of Lemma 3] First note that the the closed loop poles of the disturbance loop are the zeros of

\[ 1 + \frac{1}{(\sigma \tau + 1)^2} PC. \]

Note that the term \( \frac{1}{(\sigma \tau + 1)^2} \) does not introduce any unstable open loop poles or zeros into the feedback loop. The idea of the proof is to show that for small enough \( \tau \), the Nyquist plots of \( \frac{1}{(\sigma \tau + 1)^2} PC \) and \( PC \) encircle \(-1\) the same number of times, since that would prove that the number of unstable closed loop poles remains unchanged.

Consider a Nyquist plot of \( PC \). Since \( PC \) is strictly proper, there exists \( M > 0 \) such that \( |P(j\omega)C(j\omega)| \leq \frac{1}{2} \)
whenever \( |\omega| \geq M \). Thus all the encirclements of \(-1\) of the
Nyquist plot occur in the image of $[-jM, jM]$. Furthermore, for all $\omega$ with $|\omega| \geq M$,

$$
\frac{1}{(j\omega\tau + 1)^2} P(j\omega)C(j\omega) = \frac{1}{\omega^2\tau^2 + 1} |P(j\omega)C(j\omega)| \leq |P(j\omega)C(j\omega)| \leq \frac{1}{2}.
$$

So, similarly, all encirclements of $-1$ in the Nyquist plot of $\frac{1}{(s\tau + 1)^2} PC$ occur in the image of $[-jM, jM]$.

By internal stability and continuity of a continuous function over a compact domain, there exists $\gamma > 0$ such that $|1 + P(j\omega)C(j\omega)| \geq \gamma$ for all $\omega \in [-M, M]$. Furthermore, by continuity, $T$ can be chosen small enough such that for all $\tau \in [0, T]$, and all $\omega \in [-M, M]$,

$$
\left| 1 + \frac{1}{(j\omega\tau + 1)^2} P(j\omega)C(j\omega) \right| \geq \frac{\gamma}{2}.
$$

Thus, for all $\tau \in [0, T]$ the Nyquist plot of $\frac{1}{(s\tau + 1)^2} PC$ does not pass through $-1$. Therefore the Nyquist plots of $PC$ and $\frac{1}{(s\tau + 1)^2} PC$ must encircle $-1$ the same number of times.

REFERENCES


