Dirty Paper Coding vs. TDMA for MIMO Broadcast Channels

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Abstract—**In this paper we derive an upper bound on the sum-rate gain that dirty-paper coding provides over TDMA for MIMO broadcast channels. We find that the sum-rate capacity (achievable using dirty-paper coding) of the multiple-antenna broadcast channel is at most $\min(M, K)$ times the largest single-user capacity (i.e. the TDMA sum-rate) in the system, where M is the number of transmit antennas and K is the number of receivers. This result is independent of the number of receive antennas. We investigate the tightness of this bound in a time-varying channel (assuming perfect channel knowledge at receivers and transmitters) where the channel experiences uncorrelated Rayleigh fading and in some situations we find that the dirty paper gain is upper bounded by the ratio of transmit to receive antennas. We also show that $\min(M, K)$ upper bounds the sum rate gain of successive decoding over TDMA for the uplink, where M is the number of receive antennas at the base station and K is the number of transmitters.**

I. Introduction

In this paper we consider a broadcast channel (downlink or BC) in which there are multiple antennas at the transmitter (base station) and possibly multiple antennas at each receiver (mobile). Dirty-paper coding (DPC) [1, 2] is an exciting new transmission technique which allows a base station to efficiently transmit data to multiple users at the same time. It has recently been shown that dirty paper coding achieves the sum-rate capacity of the multiple-antenna broadcast channel [1, 3–5] and the DPC achievable region is the largest known achievable region for the multiple-antenna broadcast channel. However, dirty paper coding is a rather new and complicated scheme which has yet to be implemented in practical systems. Current systems such as Qualcomm's High Date Rate (HDR) system [6] use the much simpler technique of time-division multiple-access (TDMA) in which the base transmits to only a single user at a time. This technique achieves the sum-rate capacity when the base station has only one transmit antenna, but TDMA is suboptimal when the base station has multiple transmit antennas.

Considering the difficulty in implementing dirty-paper coding, a relevant question to ask is the following: How large of a performance boost does dirty-paper coding provide over TDMA in terms of sum-rate? Viswanathan, Venkatesan, and Huang first investigated this question by obtaining numerical results on the DPC gain in a practical, cellular setting [7]. In this paper we derive a simple analytical upper bound on the sum-rate performance gain which DPC offers over TDMA and investigate the tightness of this bound in a time-varying, Rayleigh-faded channel in which the transmitter and receiver have perfect channel knowledge. Using the same techniques, we are also able to upper bound the sum-rate gain that successive decoding provides over TDMA on the uplink (multiple-access) channel.

II. System Model

We consider a broadcast channel with K receivers, $M > 1$ transmit antennas, and $N \geq 1$ receive antennas at each receiver. Let $\mathbf{x} \in \mathbb{C}^{M\times 1}$ be the transmitted vector signal and let $\mathbf{H}_k \in \mathbb{C}^{N\times M}$ be the channel matrix of receiver k where $\mathbf{H}_k(i,j)$ represents the channel gain from transmit antenna j to antenna i of receiver k. The circularly symmetric complex Gaussian noise at receiver k is represented by $\mathbf{n}_k \in \mathbb{C}^{N\times 1}$ where $\mathbf{n}_k \sim N(0, \mathbf{I})$. Let $\mathbf{y}_k \in \mathbb{C}^{N\times 1}$ be the received signal at receiver k. The received signal is mathematically represented as

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{n}_k \qquad k = 1, \ldots, K. \tag{1}$$

The covariance matrix of the input signal is $\mathbf{\Sigma}_x \triangleq \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]$. The transmitter is subject to an average power constraint P, which implies $\mathrm{Tr}(\mathbf{\Sigma}_x) \leq P$. In the first half of this paper, we assume the channel matrix $\mathbf{H} \triangleq [\mathbf{H}_1^T \cdots \mathbf{H}_K^T]^T$ is fixed and is known perfectly at the transmitter and at all receivers. We explain the time-varying channel model in Section V.

In terms of notation, we use $\mathbf{H}^\dagger$ to indicate the conjugate transpose of matrix $\mathbf{H}$ and $||\mathbf{H}||$ to denote the matrix norm of $\mathbf{H}$, defined by $||\mathbf{H}|| = \sqrt{\lambda_{max}(\mathbf{H}^\dagger\mathbf{H})}$. We also use boldface to indicate vector and matrix quantities.

III. Sum-Rate Capacity

For the single antenna broadcast channel, sum rate capacity is achieved by transmitting to the user with the largest channel norm[1]. However, this is not generally true for a multiple transmit antenna broadcast channel. For the multiple-antenna channel, sum-rate capacity is achieved by using dirty paper coding to simultaneously transmit to several users [1, 3–5].

The expression for the sum-rate capacity of the MIMO BC in terms of the dirty paper rate region is rather complicated. However, in [3], the dirty paper rate region is shown to be equal to the capacity region of the dual MIMO multiple-access channel (MAC or uplink) with sum power constraint P, where the dual uplink is formed by changing the transmitter into an M-antenna receiver and changing each receiver into an N-antenna transmitter. The received signal in the dual MAC is given by:

$$\mathbf{y}_{MAC} = \sum_{i=1}^{K} \mathbf{H}_i^\dagger x_i + \mathbf{n} \tag{2}$$

[1]The single-antenna Gaussian broadcast channel falls into the class of *degraded* broadcast channel, for which it is known that the sum rate capacity is equal to the largest single-user capacity in the system.

where $\mathbf{H}_i^\dagger$ is the channel of each transmitter and the noise is the same as in the downlink (i.e. each component is a unit variance Gaussian). Notice that the dual channel matrix is simply the conjugate transpose of the downlink channel of each user.

Due to the MAC-BC duality, the sum-rate capacity of the MIMO BC is equal to the maximum sum-rate achievable on the dual uplink with sum power constraint P:

$$\mathcal{C}_{BC}(\mathbf{H}, P) = \max_{\{\mathbf{Q}_i:\ \mathbf{Q}_i \geq 0,\ \sum_{i=1}^K \mathrm{Tr}(\mathbf{Q}_i) \leq P\}} \log \left| \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i \right| \quad (3)$$

where each of the matrices $\mathbf{Q}_i$ is an $N \times N$ positive semi-definite covariance matrix. The expression in (3) is the sum-rate capacity of the dual uplink subject to sum power constraint P. Note that (3) is a concave maximization, for which efficient numerical algorithms exist. In this paper, we use the specialized algorithm developed in [8] for all numerical results.

The time-division rate region $\mathcal{R}_{TDMA}$ is defined as the set of average rates that can be achieved by time-sharing between single-user transmission using constant power P:

$$\mathcal{R}_{TDMA}(\mathbf{H}, P) \triangleq \left\{ (R_1, \ldots, R_K) : \sum_{i=1}^K \frac{R_i}{C_i(P, \mathbf{H}_i)} \leq 1 \right\}$$

where $C(\mathbf{H}_i, P)$ denotes the single-user capacity of the i-th user subject to power constraint P. It is easy to see that the maximum sum-rate in $\mathcal{R}_{TDMA}$ is the largest single-user capacity of the K users:

$$\mathcal{C}_{TDMA}(\mathbf{H}, P) \triangleq \max_{i=1,\ldots,K} C(\mathbf{H}_i, P). \quad (4)$$

We will refer to this quantity as the TDMA sum-rate. We define the DPC gain $G(\mathbf{H}, P)$ as the ratio of sum-rate capacity to TDMA sum-rate:

$$G(\mathbf{H}, P) \triangleq \frac{\mathcal{C}_{BC}(\mathbf{H}, P)}{\mathcal{C}_{TDMA}(\mathbf{H}, P)}. \quad (5)$$

IV. Bounds on Sum-Rate Capacity

In this section we compare the sum-rate capacity to the TDMA sum-rate. We first upper bound the sum-rate capacity of the MIMO BC, and then lower bound the TDMA sum-rate. We then use these results to upper bound the ratio of sum-rate capacity to TDMA sum-rate.

Theorem 1: The sum-rate capacity of the multiple-antenna downlink is upper-bounded by:

$$\boxed{\mathcal{C}_{BC}(\mathbf{H}, P) \leq M \log \left(1 + \frac{P}{M} ||\mathbf{H}||_{max}^2 \right)} \quad (6)$$

where $||\mathbf{H}||_{max} = \max_{i=1,\ldots,K} ||\mathbf{H}_i||$.

Proof: We prove this result using the fact that the BC sum rate capacity is equal to the dual MAC sum-rate capacity with power constraint P. The received signal in the dual MAC is $\mathbf{y}_{MAC} = \sum_{i=1}^K \mathbf{H}_i^\dagger x_i + \mathbf{n}$. The received covariance then is given by $\Sigma_y = E[\mathbf{y}\mathbf{y}^\dagger] = \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger E[x_i x_i^\dagger] \mathbf{H}_i = \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i$. Notice that the argument of the maximization in (3) is $\log |\Sigma_y|$.

The received signal power is given by $E[\mathbf{y}^\dagger \mathbf{y}] = \sum_{i=1}^K E[\mathbf{x}_i^\dagger \mathbf{H}_i^\dagger \mathbf{H}_i \mathbf{x}_i] + E[\mathbf{n}^\dagger \mathbf{n}]$. Since $\mathbf{x}^\dagger \mathbf{H}^\dagger \mathbf{H} \mathbf{x} \leq ||\mathbf{H}||^2 ||\mathbf{x}||^2$ by the definition of matrix norm, we have

$$\begin{aligned} E[\mathbf{y}^\dagger \mathbf{y}] &\leq \sum_{i=1}^K ||\mathbf{H}_i||^2 E[\mathbf{x}_i^\dagger \mathbf{x}_i] + E[\mathbf{n}^\dagger \mathbf{n}] && (7) \\ &\leq ||\mathbf{H}||_{max}^2 \sum_{i=1}^K E[\mathbf{x}_i^\dagger \mathbf{x}_i] + M && (8) \\ &\leq ||\mathbf{H}||_{max}^2 P + M && (9) \end{aligned}$$

where (8) follows from the definition of $||\mathbf{H}||_{max}$ and the fact that $E[\mathbf{n}^\dagger \mathbf{n}] = M$ and (9) follows from the sum power constraint on the transmitters in the dual MAC which implies $\sum_{i=1}^K E[\mathbf{x}_i^\dagger \mathbf{x}_i] \leq P$. Since $E[\mathbf{y}^\dagger \mathbf{y}] = Tr(E[\mathbf{y}\mathbf{y}^\dagger]) = Tr(\Sigma_y)$, this implies that $Tr(\Sigma_y) \leq P||\mathbf{H}||_{max}^2 + M$. By [9, Theorem 16.8.4], for any positive definite $M \times M$ matrix $\mathbf{K}$, $|\mathbf{K}| \leq (\frac{Tr(\mathbf{K})}{M})^M$. Therefore $|\Sigma_y| \leq (1 + \frac{P}{M}||\mathbf{H}||_{max}^2)^M$, from which we get $\mathcal{C}_{BC}(\mathbf{H}, P) = \log |\Sigma_y| \leq M \log(1 + \frac{P}{M}||\mathbf{H}||_{max}^2)$. ■

This bound is equivalent to the sum-rate capacity of a system with M spatially orthogonal eigenmodes (distributed in any manner between the K users), each with norm equal to $||\mathbf{H}||_{max}$.

Theorem 2: The TDMA sum-rate is lower bounded by the rate achieved by transmitting all power in the direction of the largest eigenmode:

$$\boxed{\mathcal{C}_{TDMA}(\mathbf{H}, P) \geq \log \left(1 + P||\mathbf{H}||_{max}^2 \right).} \quad (10)$$

Proof: For each user, $C(\mathbf{H}_i, P) \geq \log(1 + P||\mathbf{H}_i||^2)$ because single-user capacity is achieved by water-filling over *all* eigenmodes instead of allocating all power to the best eigenmode. Since the TDMA sum-rate is the maximum of the single-user capacities, the result follows directly. ■

This bound is tight when $N = 1$, but is generally not tight for $N > 1$ because each user has $\min(M, N)$ eigenmodes to water-fill over.

By combining Theorems 1 and 2, we can upper bound the DPC gain:

Theorem 3: The ratio of sum-rate capacity (achievable by DPC) to TDMA sum-rate is upper bounded by M, the number of transmit antennas.

Proof: The DPC gain is bounded as follows:

$$\begin{aligned} \frac{\mathcal{C}_{BC}(\mathbf{H}, P)}{\mathcal{C}_{TDMA}(\mathbf{H}, P)} &\leq \frac{M \log \left(1 + \frac{P}{M} ||\mathbf{H}||_{max}^2 \right)}{\log \left(1 + P||\mathbf{H}||_{max}^2 \right)} && (11) \\ &\leq M && (12) \end{aligned}$$

where we used Theorems 1 and 2 to get (11). ■

Single-user bounds on capacity imply

$$\begin{aligned} \mathcal{C}_{BC}(\mathbf{H}, P) &\leq \sum_{i=1}^K C(\mathbf{H}_i, P) \\ &\leq K \mathcal{C}_{TDMA}(\mathbf{H}, P), && (13) \end{aligned}$$

Combining Theorem 3 and (13) gives the following bound:

$$\boxed{G(\mathbf{H}, P) \leq \min(M, K).} \tag{14}$$

This bound is valid for any set of channel matrices $\mathbf{H}_1, \ldots, \mathbf{H}_K$, any number of receive antennas N, any number of users K, and any SNR P. Thus, there is the greatest potential for a large DPC gain when there are a large number of users and transmit antennas. In the next section, we actually investigate the tightness of this bound for Rayleigh-faded channels.

If we consider a system with $M \geq N$ in the regimes of high and low SNR, we get the following results (proofs of both theorems are contained in [10]):

Theorem 4: If $\mathbf{H}$ has at least $\min(M, NK)$ linearly independent rows and at least one of the channel matrices $\mathbf{H}_i$ is full row rank (i.e. has N linearly independent rows), as $P \to \infty$ we have

$$\lim_{P\to\infty} G(\mathbf{H}, P) = \min\left(\frac{M}{N}, K\right). \tag{15}$$

Theorem 5: For any $\mathbf{H}$, dirty paper coding and TDMA are equivalent at asymptotically low SNR:

$$\lim_{P\to 0} G(\mathbf{H}, P) = 1. \tag{16}$$

When $N = 1$, we conjecture that $G(\mathbf{H}, P)$ is in fact a monotonically non-decreasing function of P, but we have been unable to prove this. Interestingly, when $M = N > 1$, $G(\mathbf{H}, P)$ is generally not non-decreasing and actually achieves its maximum at a finite SNR.

Note: A bound similar to Theorem 1 for the single receive antenna ($N = 1$) downlink when users have the same channel norm and are mutually orthogonal was independently derived in an earlier paper by Viswanathan and Kumaran [11, Proposition 2].

V. Tightness of Bound in Rayleigh Fading

In this section we consider the downlink sum-rate capacity in uncorrelated Rayleigh fading, i.e. where each entry of $\mathbf{H}_k$ is distributed as a complex circularly symmetric Gaussian random variable with unit variance. Here we consider time-varying systems, but we assume the transmitter and receiver have perfect and instantaneous channel state information (CSI), and thus can adapt to the channel in each fading state. We also assume that the transmitter (the base station) is subject to a short-term power constraint, so that the base station must satisfy power constraint P in every fading state. This implies that there can be no adaptive power allocation over time.

Assuming that the fading process is ergodic, the sum-rate is equal to the expected value of the sum-rate in each fading state. By applying (14) in each fading state and taking an expectation of the sum-rate capacity and of the TDMA sum-rate, it is clear that the ratio of the average sum-rate capacity to the average TDMA sum-rate is also upper bounded by $\min(M, K)$. In this section we show that this bound can be tightened to $\min(\frac{M}{N}, K)$ in the limit of high SNR, in the limit of a large number of transmit antennas, and in the limit of a large number of users for Rayleigh fading channels.

A. High SNR

We first consider the scenario where M, N, and K are fixed, but the SNR P is taken to infinity. Furthermore, we assume $K \geq M \geq N$, which is quite reasonable for practical systems. In this scenario, the DPC gain is shown to asymptotically equal $\frac{M}{N}$. Thus, if $M = N$, then TDMA is optimal at high SNR. We show tightness of this bound by establishing upper and lower bounds on TDMA and DPC sum-rate.

Similar to Theorem 1, we can upper bound the single-user capacity $C(\mathbf{H}_i, P)$ by $N \log(1 + \frac{P}{N}||\mathbf{H}_i||^2)$. Then, using Jensen's inequality, the TDMA sum-rate can be bounded as:

$$\begin{aligned} \mathbb{E}_{\mathbf{H}}\left[\mathcal{C}_{TDMA}(\mathbf{H}, P)\right] &\leq N\mathbb{E}_{\mathbf{H}}\left[\log\left(1 + \frac{P}{N}||\mathbf{H}||_{max}^2\right)\right] \\ &\leq N\log\left(1 + \frac{P}{N}\mathbb{E}_{\mathbf{H}}[||\mathbf{H}||_{max}^2]\right). \end{aligned}$$

We can lower bound the TDMA capacity as:

$$\begin{aligned} \mathbb{E}_{\mathbf{H}}[\mathcal{C}_{TDMA}(\mathbf{H}, P)] &\geq \mathbb{E}_{\mathbf{H}_1}[C(\mathbf{H}_1, P)] \\ &\geq N\mathbb{E}_{\mathbf{H}}\left[\log\left(1 + \frac{P}{N}\lambda_i\right)\right] \\ &\geq N\left(\log\left(\frac{P}{N}\right) + \mathbb{E}_{\mathbf{H}}\left[\log\left(\lambda_i\right)\right]\right). \end{aligned}$$

where λ_i is an unordered eigenvalue of the Wishart matrix $\mathbf{H}_1^\dagger\mathbf{H}_1$ and the single-user capacity is lower bounded by transmitting equal power (as opposed to the optimal water-filling power allocation) on each of the N eigenmodes of User 1.

Using Theorem 1 and Jensen's inequality, we can upper bound the sum-rate capacity as:

$$\begin{aligned} \mathbb{E}_{\mathbf{H}}\left[\mathcal{C}_{BC}(\mathbf{H}, P)\right] &\leq M\mathbb{E}_{\mathbf{H}}\left[\log\left(1 + \frac{P}{M}||\mathbf{H}||_{max}^2\right)\right] \\ &\leq M\log\left(1 + \frac{P}{M}\mathbb{E}_{\mathbf{H}}\left[||\mathbf{H}||_{max}^2\right]\right). \end{aligned}$$

We can also lower bound the sum-rate capacity by choosing $\mathbf{Q}_i = \frac{P}{N}\mathbf{I}$ in (3) for each user:

$$\begin{aligned} \mathbb{E}_{\mathbf{H}}[\mathcal{C}_{BC}(\mathbf{H}, P)] &\geq \mathbb{E}_{\mathbf{H}}\left[\log\left|\mathbf{I} + \frac{P}{KN}\mathbf{H}^\dagger\mathbf{H}\right|\right] \\ &= M\mathbb{E}_{\mathbf{H}}\left[\log\left(1 + \frac{P}{KN}\lambda_1\right)\right] \\ &\geq M\left(\log\left(\frac{P}{KN}\right) + \mathbb{E}_{\mathbf{H}}\left[\log(\lambda_1)\right]\right) \end{aligned}$$

where λ_1 is distributed as an unordered eigenvalue of the $M \times M$ Wishart matrix $\mathbf{H}^\dagger\mathbf{H}$. Using these bounds, as P becomes large, we can upper and lower bound the ratio $\frac{\mathbb{E}_{\mathbf{H}}[\mathcal{C}_{BC}]}{\mathbb{E}_{\mathbf{H}}[\mathcal{C}_{TDMA}]}$ by $\frac{M}{N}$. It then follows that $\frac{\mathbb{E}_{\mathbf{H}}[\mathcal{C}_{BC}]}{\mathbb{E}_{\mathbf{H}}[\mathcal{C}_{TDMA}]}$ converges to $\frac{M}{N}$ in the limit of high SNR.

In Fig. 1, the ratio of sum-rate capacity to the TDMA sum-rate is plotted for a system with 20 users. The ratio is plotted for $M = 4$ and $N = 1$, $N = 2$, and $N = 4$. In each case the DPC gain converges to $\frac{M}{N}$, though it does so quite slowly for the $N = 1$ case.

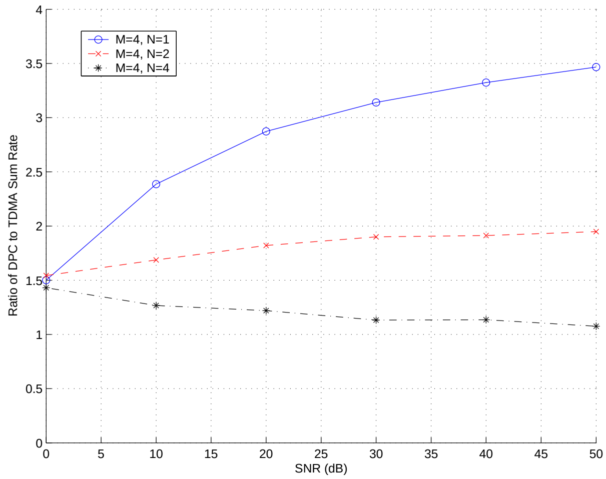

Fig. 1. DPC Gain as a function of SNR for a system with 20 users

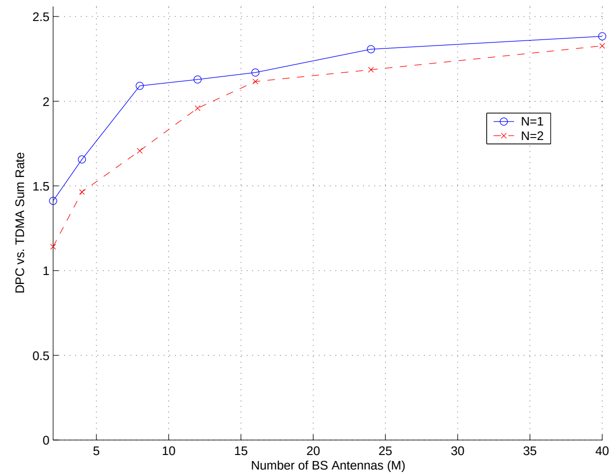

Fig. 2. DPC Gain as a function of M for a system with 3 users at 10 dB

B. Large M

In this section we examine the scenario where the number of users (K), number of receive antennas (N), and SNR (P) are fixed but the number of transmit antennas (M) is taken to be large. We will show that the DPC gain tends to K in this case.

As in the previous section, we lower bound the sum rate capacity by choosing $\mathbf{Q}_i = \frac{P}{N}\mathbf{I}$ in (3) for each user and in each fading state. The lower bound then is the point-to-point capacity of a NK transmit, M receive antenna MIMO channel in Rayleigh fading. If the number of receive antennas in this point-to-point link is allowed to become large (i.e. $M \to \infty$) but the number of transmit antennas in this point-to-point model (KN) is kept fixed, then the capacity of the point-to-point system tends to $KN\log(1+\frac{MP}{KN})$ [12].

As in the previous section, the TDMA sum-rate is upper bounded as $\mathbb{E}_{\mathbf{H}}[\mathcal{C}_{TDMA}(\mathbf{H},P)] \leq N\mathbb{E}_{\mathbf{H}}[\log\left(1+\frac{P}{N}||\mathbf{H}||^2_{max}\right)]$. Using standard probability arguments [10], we can upper bound $\mathbb{E}_{\mathbf{H}}[\mathcal{C}_{TDMA}(\mathbf{H},P)]$ by $N\log\left(1+PM(1+\alpha)\right)$, where $\alpha > 0$.

If we now take the ratio of DPC sum-rate capacity to TDMA sum-rate as $M \to \infty$, we get

$$\begin{aligned} \lim_{M\to\infty} \frac{\mathbb{E}_{\mathbf{H}}\left[\mathcal{C}_{BC}\right]}{\mathbb{E}_{\mathbf{H}}\left[\mathcal{C}_{TDMA}\right]} &\geq \lim_{M\to\infty} \frac{KN\log(1+\frac{MP}{KN})}{N\log(1+PM(1+\alpha))} \\ &= K. \end{aligned} \tag{17}$$

By (14), this ratio is also upper-bounded by K for all M, Thus, in the limit of many transmit antennas and with a fixed number of receivers, the DPC gain goes to K in the sense that the ratio of the expected value of the DPC sum-rate to the expected value of the TDMA sum-rate goes to K.

In Figure 2 the DPC gain is plotted as a function of the number of transmit antennas for a system with 3 users, each with 10 dB average SNR. Notice that for both $N=1$ and $N=2$, slow convergence to $K=3$ is observed as M becomes large.

C. Large K

If the number of antennas and the SNR are kept fixed and the number of users is taken to be large, it is shown in [13] that the dirty paper gain converges to $\frac{M}{N}$. More specifically, the authors show that the sum rate capacity and the TDMA sum-rate grow as $M\log\log(K)$ and $N\log\log(K)$, respectively.

VI. Transmitter Beamforming

Transmitter beamforming[2] is a sub-optimal technique that supports simultaneous transmission to multiple users on a broadcast channel. Each active user is assigned a beamforming direction by the transmitter and multi-user interference is treated as noise. Transmit beamforming is actually quite similar to dirty paper coding, but with DPC some multi-user interference is "pre-subtracted" at the transmitter, thus increasing the rates of some users. When $N=1$, the maximum sum rate using beamforming is given by:

$$C_{BF}(\mathbf{H},P) = \max_{\{P_i:\ \sum_{i=1}^K P_i \leq P\}} \sum_{j=1}^{K} \log \frac{\left|\mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger P_i \mathbf{H}_i\right|}{\left|\mathbf{I} + \sum_{i\neq j} \mathbf{H}_i^\dagger P_i \mathbf{H}_i\right|}$$

In [14] the authors numerically evaluate the gain that DPC provides over beamforming. Transmit beamforming actually supersedes TDMA, so an interesting open problem is to analytically bound the gain that DPC provides over transmitter beamforming.

At both asymptotically low and high SNR, beamforming performs as well as DPC in the ratio sense:

$$\lim_{P\to\infty} \frac{C_{BC}(\mathbf{H},P)}{C_{BF}(\mathbf{H},P)} = \lim_{P\to 0} \frac{C_{BC}(\mathbf{H},P)}{C_{BF}(\mathbf{H},P)} = 1. \tag{18}$$

A proof of the high SNR result is contained in [10], and the low SNR result follows from Theorem 5 and the fact that $C_{BF}(\mathbf{H},P) \geq C_{TDMA}(\mathbf{H},P)$.

Furthermore, we conjecture that the ratio $\frac{C_{BC}(\mathbf{H},P)}{C_{BF}(\mathbf{H},P)}$ is bounded by a constant ($< M$) independent of the system parameters for all P, but we are unable to prove this. Viswanathan and Venkatesan recently characterized the performance of downlink beamforming and dirty paper coding as M and K both grow to infinity at some fixed ratio $\frac{M}{K} = \alpha$. In this asymptotic regime,

[2]Transmitter beamforming is also referred to as SDMA, or space-division multiple access.

the ratio $\frac{C_{BC}(H,P)}{C_{BF}(H,P)}$ is in fact bounded by 2 for all values of α and P.

VII. Bound on Sum-Rate Gain of Successive Decoding for Uplink

Successive decoding is a capacity-achieving scheme for the multiple-access channel (uplink) in which multiple users transmit simultaneously to the base station and the receiver successively decodes and subtracts out the signals of different users. This technique achieves the sum-rate capacity of the MIMO MAC [9, Chapter 14], but is difficult to implement in practice. A sub-optimal transmission scheme is to allow only one user to transmit at a time. Using the proof technique of Theorem 1 on the dual uplink (K transmitters with N antennas each and a single receiver with M antennas) along with the individual power constraints $\mathbf{P} = (P_1, \ldots, P_K)$ on the MAC, it can be shown that the following bound holds:

$$\begin{aligned} \mathcal{C}_{MAC}(\mathbf{H}, \mathbf{P}) &= \max_{\{\mathrm{Tr}(\mathbf{Q}_i) \le P_i \ \forall i\}} \log \left| \mathbf{I} + \sum_{i=1}^{K} \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i \right| \\ &\le M \log \left(1 + \frac{\sum_{i=1}^{K} P_i ||\mathbf{H}_i||^2}{M} \right). \qquad (19) \end{aligned}$$

Notice that the sum-rate capacity of the MAC is identical to the BC sum-rate capacity expression in (3) except that the MAC expression has *individual* power constraints instead of a sum constraint.

The TDMA region for the uplink is defined differently than for the downlink because each transmitter in the uplink is subject to an average power constraint:

$$\mathcal{R}_{TDMA}(\mathbf{H}, P_1, \ldots, P_K) \triangleq \bigcup_{\alpha_i \ge 0,\ \sum_{i=1}^{K} \alpha_i = 1} \left(\alpha_1 C\left(\mathbf{H}_1, \frac{P_1}{\alpha_1}\right), \ldots, \alpha_K C\left(\mathbf{H}_K, \frac{P_K}{\alpha_K}\right) \right).$$

The TDMA sum-rate is then defined to be the maximum sum of rates in this region. As used in the proof of Theorem 2, for each user we have $C(\mathbf{H}_i, \frac{P_i}{\alpha_i}) \ge \log(1 + \frac{P_i}{\alpha_i}||\mathbf{H}_i||^2)$ for any α_i. Thus,

$$\mathcal{C}_{TDMA}(\mathbf{H}, \mathbf{P}) \ge \max_{\alpha_i \ge 0,\ \sum_{i=1}^{K} \alpha_i = 1} \sum_{i=1}^{K} \alpha_i \log \left(1 + \frac{P_i}{\alpha_i} ||\mathbf{H}_i||^2 \right).$$

The RHS of this expression corresponds to the TDMA region of a *scalar* MAC with channel gains $||\mathbf{H}_1||, \ldots, ||\mathbf{H}_K||$. It is easy to verify this expression is maximized by choosing $\alpha_i = \frac{P_i ||\mathbf{H}_i||^2}{\sum_{j=1}^{K} P_j ||\mathbf{H}_j||^2}$. We then get the following upper bound:

$$\mathcal{C}_{TDMA}(\mathbf{H}, \mathbf{P}) \ge \log \left(1 + \sum_{i=1}^{K} P_i ||\mathbf{H}_i||^2 \right). \qquad (20)$$

Combining (19) and (20) we get $\frac{\mathcal{C}_{MAC}(\mathbf{H},\mathbf{P})}{\mathcal{C}_{TDMA}(\mathbf{H},\mathbf{P})} \le M$. As before, the single-user capacity of each user also upper bounds this ratio by K. Thus, we finally get

$$\boxed{\frac{\mathcal{C}_{MAC}(\mathbf{H}, \mathbf{P})}{\mathcal{C}_{TDMA}(\mathbf{H}, \mathbf{P})} \le \min(M, K)} \qquad (21)$$

or that performing optimal successive decoding at the base station offers a gain of at most $\min(M, K)$ over TDMA.

VIII. Conclusion

We have found that the performance gain of DPC versus TDMA is upper-bounded by $\min(M, K)$, where M is the number of transmit antennas (at the base station) and K is the number of users. This bound applies at any SNR and for any number of receive antennas. For Rayleigh fading channels, the bound tightens to $\min(\frac{M}{N}, K)$ at high SNR, for a large number of transmit antennas, or for a large number of users. Using the same techniques for the uplink, we found that the performance gain using successive decoding on the uplink versus TDMA is also upper bounded by $\min(M, K)$, where M is the number of receive antennas (at the base station) and K is the number of mobiles (i.e. transmitters). Thus, it seems that for systems with many users, significant gains can be achieved by adding additional base station antennas. However, if the number of mobile antennas is the same as the number of base station antennas, the benefit of using DPC on the downlink or successive decoding on the uplink may be limited.

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