High SNR Analysis of MIMO Broadcast Channels

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***Abstract*— The behavior of the multiple antenna broadcast channel at high SNR is investigated. The multiple antenna broadcast channel achieves the same multiplexing gain as the system in which all receivers are allowed to perfectly cooperate (i.e. transforming the system into a point-to-point MIMO system). However, the multiplexing gain alone is not sufficient to accurately characterize the behavior of sum rate capacity at high SNR. An affine approximation to capacity which incorporates the multiplexing gain as well as a power offset (i.e. a zero-order term) is a more accurate representation of high SNR behavior. The power offset of the sum rate capacity is shown to equal the power offset of the cooperative MIMO system when there are less receivers than transmit antennas. In addition, the power offset of using the sub-optimal strategy of beamforming is calculated. These calculations show that beamforming can perform quite well when the number of antennas is sufficiently larger than the number of receivers, but performs very poorly when there are nearly as many receivers as transmit antennas.**

I. Introduction

We consider a multiple antenna broadcast channel (BC), in which the transmitter has multiple antennas and each of the receivers has a single antenna. It is now well known that dirty paper coding (DPC) achieves the sum capacity of the multiple antenna broadcast channel as well as the full capacity region [12] [1] [10] [11] [13]. Though this channel has been extensively studied, there is no general closed form expression for the sum capacity, and it can be expressed in general only as the solution to a convex optimization problem.

In this paper, we investigate the high SNR behavior of the sum capacity of the multiple antenna broadcast channel. We study the sum rate capacity (achieved using DPC) as well as the sum rate achievable using linear beamforming (without DPC). It is well known that the sum rate capacity of the MIMO BC achieves the same multiplexing gain as a system where all receivers are allowed to cooperate. However, the multiplexing gain, which quantifies the rate of increase of capacity as a function of the SNR, is not sufficient to capture all relevant high SNR features.

A more accurate representation of high SNR behavior is provided by an affine approximation to capacity, which includes both the multiplexing gain (i.e. slope) as well as a power offset (i.e. zero-order term). Such an affine approximation is useful because: (i) it is able to capture the non-negligible effect of different fading models and increasing the number of transmit antennas beyond the number of receive antennas, or vice versa, which the multiplexing gain does not reflect, and (ii) the high SNR approximation provides accurate results for even moderate SNR values, and thus is of practical interest [8]. This approximation was first developed for randomly spread CDMA channels in [5], and has been applied to different CDMA models [7] [9] as well as point-to-point MIMO channels [8] [4].

We consider MIMO downlink channels in which the number of receivers is no larger than the number of transmit antennas. In this scenario, we show that the affine approximation to the sum rate capacity is identical to the affine approximation of the cooperative MIMO channel, which is the resultant point-to-point channel when all receivers are allowed to cooperate. In addition, we study the maximum sum rate achievable using beamforming, and provide closed form expressions for the power offset relative to the true sum rate capacity.

II. System Model

We consider an M transmit antenna, single receive antenna broadcast channel with K receivers, with $K \leq M$. The assumption on the number of users is crucial to this work. The broadcast channel is mathematically described as:

$$\mathbf{y}_i = \mathbf{h}_i \mathbf{x} + \mathbf{n}_i, \quad i = 1, \ldots, K \tag{1}$$

where $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_K$ are the channel vectors (with $\mathbf{h}_i \in \mathbb{C}^{1 \times M}$) of users 1 through K respectively on the downlink, the vector $\mathbf{x} \in \mathbb{C}^{M \times 1}$ is the downlink transmitted signal, and $\mathbf{n}_1, \ldots, \mathbf{n}_K$ are independent complex Gaussian noise terms with unit variance. There is a transmit power constraint of P, i.e. the input must satisfy $E[||\mathbf{x}||^2] \leq P$.

We denote the concatenation of the channels by $\mathbf{H}^\dagger = [\mathbf{h}_1^\dagger \mathbf{h}_2^\dagger \cdots \mathbf{h}_K^\dagger]$, i.e. $\mathbf{H}$ is $K \times M$ with the i-th row equal to the channel of the i-th receiver. We consider a slowly fading channel, and assume the transmitter and receivers have perfect and instantaneous channel knowledge.

III. High SNR Approximation

In [5], the following approximation to capacity was proposed for asymptotically large SNR:

$$\begin{aligned} C(P) &= \mathcal{S}_\infty \left(\log_2(P) - \mathcal{L}_\infty \right) + o(1) \\ &= \mathcal{S}_\infty \left(\frac{P_{dB}}{3dB} - \mathcal{L}_\infty \right) + o(1), \end{aligned} \tag{2}$$

where $\mathcal{S}_\infty$ represents the multiplexing gain, $\mathcal{L}_\infty$ represents the power offset (in 3 dB units), and the $o(1)$ term vanishes as

$P \to \infty$. The multiplexing gain $\mathcal{S}_\infty$ and the power offset $\mathcal{L}_\infty$ are defined as:

$$\mathcal{S}_\infty = \lim_{P\to\infty} \frac{C(P)}{\log_2(P)}, \quad (3)$$

$$\mathcal{L}_\infty = \lim_{P\to\infty} \left(\log_2(P) - \frac{C(P)}{\mathcal{S}_\infty} \right). \quad (4)$$

The capacity achieving strategy of dirty paper coding and the sub-optimal technique of beamforming both achieve a multiplexing gain of $\min(M, K)$. In the remainder of this paper, we characterize the power offset term $\mathcal{L}_\infty$ for the sum rate capacity and the beamforming sum rate.

IV. Sum Rate Capacity

The sum rate capacity of the MIMO BC (denoted $\mathcal{C}_{BC}(P, \mathbf{H})$) is given by the following expression [10]:

$$\mathcal{C}_{BC}(P, \mathbf{H}) = \max_{\{P_i\}_{i=1}^K: \ \sum_{i=1}^K P_i \le P} \log \left| \mathbf{I} + \sum_{i=1}^K P_i \mathbf{h}_i^\dagger \mathbf{h}_i \right|, \quad (5)$$

where the maximization is performed over powers $P_1, \ldots, P_K$. This expression is the capacity of the dual MIMO multiple-access channel, which is equal to the capacity of the MIMO broadcast channel [10].

When $K = 2$, a closed-form expression for this maximization can be calculated [1]. In addition, the optimum power policy that maximizes the dual MAC expression in (5) has a waterfilling interpretation, in that it is optimal to allocate all power to the user with the larger channel norm up to some point, after which any additional power should be split evenly between both users. When $K > 2$, however, no closed form expression exists for sum rate capacity. In addition, by examining the KKT conditions that characterize the solution to (5), we can show that the optimum power allocation does not follow the waterfilling form, though it is numerically found to be extremely close to waterfilling. However, we are able to show that (5) has a unique solution:

Theorem 1: There exists a unique solution to the sum rate capacity maximization in (5) if $\{\mathbf{h}_i^\dagger \mathbf{h}_i\}_{i=1}^K$ are linearly independent. Furthermore, if the vectors $\{\mathbf{h}_i\}_{i=1}^K$ are linearly independent, then the outer products are independent as well.

Proof: See [3]. ∎

Let us now move on to the calculation of the power offset $\mathcal{L}_\infty$ for the sum rate capacity. In order to do so, we first define the capacity of the point-to-point MIMO system with channel matrix $\mathbf{H}$:

$$\mathcal{C}_{MIMO}(P, \mathbf{H}) = \max_{\mathbf{Q} \ge 0, \ \mathrm{Tr}(\mathbf{Q}) \le P} \log \left| \mathbf{I} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger \right|. \quad (6)$$

This capacity is referred to as the *cooperative* upper bound, because it is the capacity if all K receivers are allowed to perfectly cooperate. Clearly we have $\mathcal{C}_{MIMO}(P, \mathbf{H}) \ge \mathcal{C}_{BC}(P, \mathbf{H})$, since receiver cooperation can only help performance. Note that the reciprocity of MIMO capacity implies $\mathcal{C}_{MIMO}(P, \mathbf{H}) = \mathcal{C}_{MIMO}(P, \mathbf{H}^\dagger)$ [6]. The term $\mathcal{C}_{MIMO}(P, \mathbf{H})$ is the capacity of the point-to-point MIMO capacity when the transmitter is given channel state information (CSI) and is allowed to optimize its input to the channel (i.e. waterfill along the eigenvalues of the channel). It is often simpler to study the mutual information achieved using an isotropic input. We are interested in the mutual information of the reciprocal channel (i.e. $\mathbf{H}^\dagger$), which is defined as:

$$\mathcal{I}_{MIMO}(P, \mathbf{H}^\dagger) = \log \left| \mathbf{I} + \frac{P}{K} \mathbf{H}^\dagger \mathbf{H} \right|. \quad (7)$$

Note that channel reciprocity does not hold for the mutual information, and we in fact have (with probability one) $\mathcal{I}_{MIMO}(P, \mathbf{H}^\dagger) > \mathcal{I}_{MIMO}(P, \mathbf{H})$ when $K < M$ and $\mathbf{H}$ is iid Rayleigh distributed.

Theorem 2: At high SNR, the sum rate capacity of the MIMO BC with $M \ge K$ and concatenated channel matrix $\mathbf{H}$ behaves identically to the mutual information achieved with an isotropic input of the $K \times M$ point-to-point MIMO channel with gain matrix $\mathbf{H}^\dagger$, in the sense that both capacities have the same affine high-SNR approximation (as defined in (2)).

Proof: This result essentially follows from Theorem 3 of [1], which shows that if $\mathbf{H}$ is full row rank, then:

$$\lim_{P\to\infty} \left[\mathcal{C}_{MIMO}(P, \mathbf{H}) - \mathcal{C}_{BC}(P, \mathbf{H}) \right] = 0.$$

The reciprocity of MIMO channels gives $\mathcal{C}_{MIMO}(P, \mathbf{H}) = \mathcal{C}_{MIMO}(P, \mathbf{H}^\dagger)$, and we also have

$$\lim_{P\to\infty} \left[\mathcal{C}_{MIMO}(P, \mathbf{H}^\dagger) - \mathcal{I}_{MIMO}(P, \mathbf{H}^\dagger) \right] = 0$$

since waterfilling leads to a vanishing increase in capacity at asymptotically high SNR when there are more receive antennas than transmit antennas (where K and M are the numbers of transmit and receive antennas, respectively, in the *reciprocal* channel). Thus the sum rate capacity of the MIMO BC and the mutual information of the reciprocal MIMO channel have the same asymptotic (in SNR) behavior for each instantiation of $\mathbf{H}$. This equivalence also holds in the expected value sense, i.e. $E[\mathcal{C}_{BC}(P, \mathbf{H})]$ and $E[\mathcal{I}_{MIMO}(P, \mathbf{H}^\dagger)]$ are asymptotically equal. ∎

As a result of Theorem 2, the power offset for the sum rate capacity of the MIMO BC is equal to the power offset of the mutual information of the $K \times M$ point-to-point MIMO capacity, a number of which are known in closed form.

To gain some intuition, consider the affine approximation of the sum rate capacity for a specific $\mathbf{H}$:

$$\begin{aligned} \mathcal{C}_{BC}(P, \mathbf{H}) &\cong \mathcal{C}_{MIMO}(P, \mathbf{H}) \\ &\cong \sum_{i=1}^K \log_2 \left(\frac{P}{K} \lambda_i \right) \\ &= K \log_2(P) + \log_2 |\mathbf{H}\mathbf{H}^\dagger| - K \log_2 K \end{aligned}$$

where $\lambda_1, \ldots, \lambda_K$ denote the eigenvalues of the $K \times K$ Wishart matrix $\mathbf{H}\mathbf{H}^\dagger$, and $\cong$ refers to equivalence in the limit (i.e. the difference between both sides converges to zero as

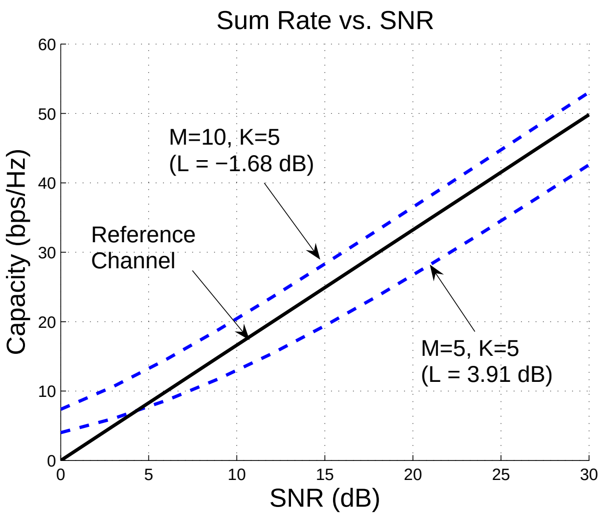

Fig. 1. High SNR Behavior in Rayleigh Fading

$P \to \infty$). Thus the power offset term for a specific $\mathbf{H}$ is given by:

$$\mathcal{L}_\infty(\mathbf{H}) = \log_2 K - \frac{1}{K} \log_2 |\mathbf{H}\mathbf{H}^\dagger|. \quad (8)$$

When considering fading channels, the power offset is given by the expected value of $\mathcal{L}_\infty(\mathbf{H})$. This term has in fact been computed in closed-form (in the context of point-to-point MIMO channels) for the iid Rayleigh distribution ([8] and references therein), Rayleigh with antenna correlation [4], and for the Ricean distribution [8]. In Rayleigh fading, for example, the power offset in 3-dB units is [8]:

$$\mathcal{L}_\infty^{iid} = \log_2 K + \left(\gamma + 1 - \sum_{l=1}^{M-K} \frac{1}{l} - \frac{M}{K} \sum_{l=M-K+1}^{M} \frac{1}{l} \right) \log_2 e$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant.

In Fig. 1, average sum capacity is plotted for systems with $M = 5, K = 5$ and $M = 10, K = 5$ and iid Rayleigh fading. In addition, the reference curve, with a slope of 5 bps/Hz/ 3 dB, is also plotted. Notice that both systems have a multiplexing gain of 5, but there is a power shift of approximately 5.6 dB. Increasing the number of transmit antennas does not change the multiplexing gain, but has a significant effect on the power offset term.

Using Theorem 2, we are also able to show that power optimization is not asymptotically necessary when using DPC:

Corollary 1: The difference between the sum rate capacity and the sum rate achieved with equal power allocation in (5) converges to zero.

$$\lim_{P \to \infty} \left[\mathcal{C}_{BC}(P, \mathbf{H}) - \log \left| \mathbf{I} + \sum_{i=1}^{K} \frac{P}{K} \mathbf{h}_i \mathbf{h}_i^\dagger \right| \right] = 0 \quad (9)$$

Proof: See [3]. ∎

Note that this corollary refers to using equal power on the dual MAC, and not directly on the downlink channel. The downlink covariance matrices are related to the dual MAC powers by the transformations in [10] [11].

Though we focus on single antenna receivers in this work, Theorem 2 applies to MIMO downlink channels with multiple receive antennas if the total number of receiver antennas is no larger than M. For example, if each receiver has N antennas and $NK \leq M$, then the theorem applies. This implies that such a channel behaves identical to a $NK \times M$ MIMO channel. Thus, the high SNR capacity only depends on the product N and K and not on their specific values; for example, if the transmitter has 8 antennas, then a system with 8 single-antenna receivers is equivalent to a system with four dual-antenna receivers.

V. Beamforming Sum Rate

We now analyze the performance of transmit beamforming, and quantify the loss relative to DPC at high SNR. Transmit beamforming differs from DPC in that no interference is pre-cancelled at the transmitter. As a result, each user experiences interference from every other user. The dual MAC expression for the beamforming capacity (denoted $\mathcal{C}_{BF}(P, \mathbf{H})$) is:

$$\mathcal{C}_{BF}(P, \mathbf{H}) = \max_{\{P_i:\ \sum_{i=1}^K P_i \leq P\}} \sum_{j=1}^{K} \log \frac{\left| \mathbf{I} + \sum_{i=1}^K P_i \mathbf{h}_i^\dagger \mathbf{h}_i \right|}{\left| \mathbf{I} + \sum_{i \neq j} P_i \mathbf{h}_i^\dagger \mathbf{h}_i \right|} \quad (10)$$

In order to maximize rates when using beamforming, the transmit beamformers (in the downlink) or the receive beamvectors (in the dual uplink) should be chosen using the MMSE criterion.

In order to make this comparison, we also consider the rates achievable using zero-forcing beamforming. When zero-forcing is used, the beamforming directions are chosen such that each receiver experiences no multi-user interference. The maximum sum rate achievable using zero-forcing is given by:

$$\mathcal{C}_{ZF}(P, \mathbf{H}) = \max_{\{P_i:\ \sum_{i=1}^K P_i \leq P\}} \sum_{j=1}^{K} \log\left(1 + P_i ||\mathbf{g}_i||^2\right) \quad (11)$$

where $\mathbf{g}_i$ is the orthogonal complement of $\mathbf{h}_i$ with respect to the space spanned by $\{\mathbf{h}_j\}_{j \neq i}$ (i.e. the projection of $\mathbf{h}_i$ onto the nullspace of span($\{\mathbf{h}_j\}_{j \neq i}$)). Notice that we have $||\mathbf{g}_i||^2 = 1/[(\mathbf{H}\mathbf{H}^\dagger)^{-1}]_{ii}$.

The following theorem proves that the difference between transmit beamforming using MMSE vectors (i.e. rates of the form in (10)) and zero-forcing beamforming converges to zero. Note that this is a formal statement of the well known fact that an MMSE receiver converges to a zero-forcing receiver at high SNR.

Theorem 3: The rates achievable using zero-forcing beamforming and MMSE beamforming using any power allocation of the form $P_i = \alpha_i P$ converge at asymptotically high SNR:

$$\lim_{P\to\infty}\left[\log\frac{\left|\mathbf{I}+\sum_{i=1}^{K}\alpha_i P\mathbf{h}_i^{\dagger}\mathbf{h}_i\right|}{\left|\mathbf{I}+\sum_{i\neq j}\alpha_i P\mathbf{h}_i^{\dagger}\mathbf{h}_i\right|}\right.$$
$$\left.-\sum_{j=1}^{K}\log\left(1+\alpha_j P||\mathbf{g}_j||^2\right)\right]=0,$$

for $j = 1, \ldots, K$.

Proof: See [3] ■

This theorem proves the equivalence of MMSE beamforming and zero-forcing beamforming for each power allocation. It is intuitively clear that the same statement holds if the best power allocation is considered for both strategies, but there are some mathematical technicalities that make rigorously proving this difficult. Thus, though the proceeding theorem is for zero-forcing beamforming, it is understood to hold for optimal transmit beamforming as well.

Theorem 4: The use of zero-forcing leads to a loss of $\log_2 \frac{|\mathbf{H}\mathbf{H}^{\dagger}|}{\prod_{i=1}^{K}||\mathbf{g}_i||^2}$ bps/Hz in spectral efficiency relative to DPC.

Proof: Since waterfilling has no effect at high SNR, we have $\mathcal{C}_{ZF}(P,\mathbf{H}) \cong \sum_{i=1}^{K}\log\left(\frac{P}{K}||\mathbf{g}_i||^2\right)$, while $\mathcal{C}_{BC}(P,\mathbf{H}) \cong \log\left(\frac{P}{K}|\mathbf{H}\mathbf{H}^{\dagger}|\right)$. Thus we have the result. ■

We denote the loss term as $\beta_{ZF} \triangleq \log_2 \frac{|\mathbf{H}\mathbf{H}^{\dagger}|}{\prod_{i=1}^{K}||\mathbf{g}_i||^2}$. Thus, the sum rate achieved using ZF can be approximated at high SNR as:

$$\mathcal{C}_{ZF}(P,\mathbf{H}) \approx \mathcal{C}_{BC}(P,\mathbf{H}) - \beta_{ZF}. \tag{12}$$

When $\mathbf{H}$ is iid Rayleigh, a simple expression for the expected loss can be found.

Theorem 5: The expected loss in Rayleigh fading due to beam-forming is given by:

$$E_{\mathbf{H}}[\beta_{ZF}] = \log_2 e \sum_{j=1}^{K-1}\frac{j}{M-j} \quad \text{bps/Hz}, \tag{13}$$

which corresponds to a power offset of

$$\frac{3\log_2 e}{M}\sum_{j=1}^{K-1}\frac{j}{M-j} \quad \text{dB}. \tag{14}$$

Proof: The result follows from the fact that $||\mathbf{g}_i||^2$ is chi-square with two degrees of freedom for all i, while $|\mathbf{H}\mathbf{H}^{\dagger}|$ is the product of K chi-square rv's with $2M, 2(M-1), \ldots, 2(M-K+1)$ degrees of freedom. ■

Furthermore, when $M = K$, we have

$$E_{\mathbf{H}}[\beta_{ZF}] \approx M\log_2 M \tag{15}$$

in the sense that the ratio of both sides converges to one as M grows large. This corresponds to a power offset of $3\log_2 M$ dB, which can be extremely large. Note that the approximation $3\log_2 M$ dB overstates the power penalty by 1 - 1.5 dB for reasonable values of M (< 20), but does capture the growth rate. Such a large penalty is not surprising, since the use of zero-forcing requires inverting the $M \times M$ matrix $\mathbf{H}$, which is poorly conditioned with high probability when M is large.

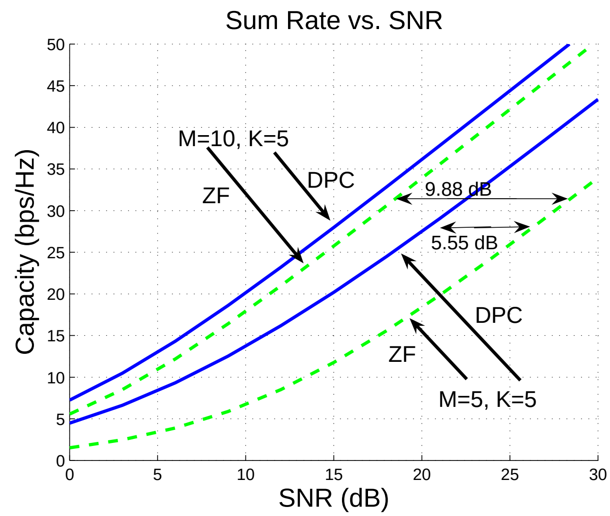

Fig. 2. DPC vs. Zero-Forcing at High SNR

These results are quite insightful as they give the loss incurred by using beamforming without interference cancellation relative to using DPC in a very simple form. In a five transmit antenna, five receiver system for example, the expected power penalty of beamforming is 5.55 dB (whereas the approximation in (15) gives 6.97 dB), which is rather significant. This gap is shown in Fig. 2, along with curves for a $M = 10, K = 5$ system. For this system, the penalty for using zero-forcing is only 1.26 dB. Increasing the number of transmit antennas from 5 to 10 shifts the sum rate capacity curve by 5.59 dB, but improves the performance of zero-forcing by 9.88 dB. This is because zero-forcing gains the increase in the $\mathcal{L}_\infty$ term of sum-rate capacity (5.59 dB, also seen in Fig. 1), along with the significantly decreased zero-forcing penalty due to the increased number of transmit antennas (5.55 dB to 1.26 dB). Thus adding transmit antennas has the dual benefit of increasing the performance of dirty paper coding, as well as significantly decreasing the gap between the computationally simpler technique of zero-forcing beamforming and DPC.

If the number of users and transmit antennas are taken to infinity at a fixed ratio (i.e. $M = \alpha K$ with $K \to \infty$) with $\alpha > 1$, then the power offset between DPC and zero-forcing is found to be bounded. For example, if $\alpha = 2$, or the number of transmit antennas is double the number of receivers, the zero-forcing penalty is no larger than 1.67 dB, and monotonic convergence to this asymptote is observed. Table I contains the asymptotic power penalty for different values of α. Note that the penalty for any finite M is no larger than the asymptotic value. Thus for large systems, zero-forcing is a viable low-complexity alternative to DPC if the number of transmit antennas can be made suitably large. A similar conclusion was drawn in [2] where the ratio of the rates achievable with zero-forcing relative to the sum rate capacity

TABLE I

ASYMPTOTIC BEAMFORMING PENALTY ($M = \alpha K, M \to \infty$)

α	Power Penalty
1	∞ ($\approx 3 \log_2 M$)
1.1	7.1 dB
1.33	3.7 dB
1.5	2.8 dB
2	1.7 dB
3	0.9 dB

is studied. It should also be noted that using zero-forcing and linear MMSE beamforming on the downlink channel is nearly identical to using a decorrelating receiver or a linear MMSE receiver, respectively, for an uplink MIMO or randomly spread CDMA channel. See [5] for similar asymptotic CDMA results.

The expected power penalty of beamforming under non-iid fading models is also of interest. As mentioned earlier, closed-form expressions for the MIMO power offset for Ricean fading and antenna correlation are known. In order to derive such expressions, one must compute the expected value of the matrix $\mathbf{H}\mathbf{H}^\dagger$. To compute β_{ZF}, however, one must perform computations on the inverse of this matrix, which are in general difficult. As a result, the beamforming power offset for either Ricean fading or antenna correlation are still unknown.

In Ricean fading, the effect of the line-of-sight (LOS) component on the gap between beamforming and DPC strongly depends on the specific LOS matrix. If the LOS matrix has rows that are roughly orthogonal, which implies that beamforming and DPC are nearly equivalent on the LOS matrix, then a larger K factor will decrease the gap between beamforming and DPC; as $K \to \infty$, the channel is deterministically equal to the mean matrix, for which there is only a small gap between DPC and beamforming, by assumption. If, on the other hand, the mean matrix is itself poorly suited for beamforming (i.e. the rows are very far from orthogonal to each other, corresponding to a large β_{ZF} term), then we would expect the beamforming power offset to increase with the K factor. Thus, no simple explanation for the effect of Ricean fading on beamforming can be given. Notice the same statement is also true for MIMO capacity: certain LOS components increase capacity, while others decrease capacity [8].

It is also difficult to ascertain the effect of antenna correlation on beamforming. Since receivers are not co-located in a MIMO downlink channel, only the transmit antennas can be correlated. Transmit correlation implies that the columns of the concatenated matrix $\mathbf{H}$ are correlated. This might intuitively seem to decrease the orthogonality of the rows of $\mathbf{H}$, thereby making beamforming more sub-optimal, but this has not been analytically verified.

VI. CONCLUSION

In this paper we studied the high SNR behavior of the multiple antenna broadcast channel. We showed that the sum rate capacity of a multiple antenna broadcast channel (with more transmit antennas than receivers) behaves asymptotically identically to the cooperative point-to-point MIMO system at high SNR. Thus, we were able to directly apply all high-SNR MIMO results to the downlink channel. In addition, we studied the rates achievable using linear beamforming. We computed closed-form expressions for the power offset of beamforming relative to sum rate capacity, and found that beamforming is a viable alternative to dirty paper coding only if the number of transmit antennas is substantially higher than the number of receivers.

An obvious extensions of this work is to analyze MIMO downlink channels with more receivers than transmit antennas. In this regime, however, no simple equivalence to the cooperative point-to-point MIMO channel can be made. As a result, a completely different approach is necessary to characterize the power offset of such channels, and preliminary results do indeed show significant differences from the scenario analyzed in this paper. For example, the number of antennas per receiver does affect the power offset when there are more aggregate receive antennas than transmit antennas; in the scenario analyzed in this paper, only the aggregate number of receive antennas is of importance.

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