# On the Duality between General Multiple-Access/Broadcast Channels

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*Abstract* — **We investigate expanding the recently established Gaussian multiple-access/broadcast channel duality [1] to a duality between discrete memoryless broadcast and multiple-access channels. Analogous to the Gaussian case, it is interesting to ask if the capacity region of a DM broadcast channel can be written as a union of dual MAC capacity regions (where the union is over a set of dual MAC channels). We make some preliminary progress towards answering this question.**

We consider a two-user discrete memoryless (DM) broadcast channel consisting of an input alphabet $\mathcal{X}$ of cardinality $M$, output alphabets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ each of cardinality $N$, and a probability transition function $p(y_1, y_2|x)$. Similarly, we consider the *dual* DM multiple-access channel consisting of input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$ each of cardinality $N$, an output alphabet $\mathcal{Y}$ of cardinality $M$, and a probability transition function $p(y|x_1, x_2)$. For deterministic channels, the output is a deterministic function of the input, and thus we represent the BC transition function by $(y_1, y_2) = f(x)$ and the MAC transition function by $y = f(x_1, x_2)$.

We first establish a theorem relating the capacity regions of the deterministic BC and MAC.

**Theorem 1** *When $M = aN$, the convex hull of the union of capacity regions of the BC equals the convex hull of the union of capacity regions of the MAC*

$$Co\left(\bigcup_{f(x)} \mathcal{C}_{BC}(f(x))\right) = Co\left(\bigcup_{f(x_1,x_2)} \mathcal{C}_{MAC}(f(x_1,x_2))\right), \tag{1}$$

*where the union is taken over all deterministic channel functions for the BC and MAC.*

It is easy to show that Theorem 1 also holds for the case where $M = 3$ and $N = 2$ (i.e. the Blackwell channel[2]). Figure 1 illustrates the capacity regions of the BC and MAC for this channel. Notice that the convex hull of the two MAC capacity regions equals the capacity region of the BC (no convex hull is needed for the BC). We conjecture Theorem 1 holds for $N < M < 2N$, but we have been unable to prove this. However, we can show that this theorem does not extend to general $M$ and $N$ by way of a counterexample for $M = 8$ and $N = 3$.

We are also able to establish an upper bound to the capacity region of a DM broadcast channel by considering a related finite-state deterministic BC (i.e. the channel is deterministic in each state):

**Theorem 2** *The capacity region of a DM broadcast channel is upper bounded by the intersection of capacity regions of finite-state deterministic BC's where the transmitter and receivers know the state:*

$$\mathcal{C}_{BC} \subseteq \bigcap_{p(s),\ p(y_1,y_2|x,s)} \mathcal{C}_{BC}(p(s), p(y_1,y_2|s)) \tag{2}$$

*where the intersection is taken over all $p(y_1, y_2|x, s)$ such that the channel $p(y_1, y_2|x, s)$ is deterministic for each $s$ and that satisfy $\sum_{i=1}^{k} p(y_1, y_2|x, s = i)p(s = i) = p(y_1, y_2|x)$. Here we use $\mathcal{C}_{BC}(p(s), p(y_1, y_2|x, s))$ to indicate the capacity region of the finite-state deterministic channel with transmitter and receiver knowledge of the state, given by:*

$$\mathcal{C}_{BC}(p(s), p(y_1,y_2|x,s)) = p_1 C(s=1) + \cdots + p_k C(s=k) \tag{3}$$

*where $C(s = k)$ is the capacity region of the deterministic broadcast channel when $s = k$.*

This bound implies that the capacity region of any non-deterministic channel lies within the convex hull of the capacity regions of a set of deterministic channels. Theorem 2 also applies to the MAC. Therefore, the union in Theorem 1 can be generalized to include all deterministic and non-deterministic channels. Thus, for $M = aN$, it may be possible that the capacity region of a given broadcast channel is equal to a union of MAC capacity regions, for some yet unknown dual MAC channels. However, our counter-example for $M = 8$ and $N = 3$ indicates that there is at least one broadcast channel for which this is not possible.

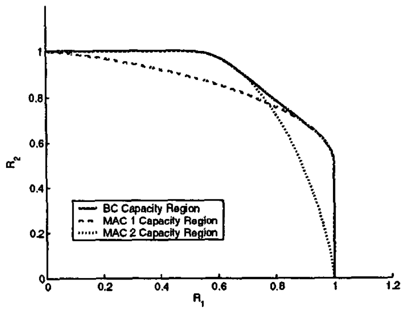


Figure 1: Deterministic BC and MAC capacity regions for $M = 3$, $N = 2$

## REFERENCES

[1] N. Jindal, S. Vishwanath, and A. Goldsmith, "On the Duality of Multiple-Access and Broadcast Channels", *Allerton Conference on Commun., Control, and Computing*, Oct. 2001. Also submitted to IEEE Trans. on Inform. Theory.

[2] S.I. Gel'fand, "Capacity of one broadcast channel," translated in Probl. Inform. Transm., pp. 92-102, Apr-June 1978.

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