Feedback capacity of multiple access & broadcast channels

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Outline

- Single-user capacity with feedback
- Multiple-access channel (MAC)
  - Achievable region of Cover & Leung
    - Pf: Block Markov Coding (Random Coding)
  - Capacity not known – an outer bound
  - White Gaussian MAC
    - For 2 Tx capacity is found by Ozarow
      - Pf: Deterministic coding of Schalkwijk & Kailath
Outline 2

- Broadcast Channel (BC)
  - Physically degraded BC
    - No capacity increase with feedback
  - Stochastically degraded BC
    - White Gaussian BC
      - Achievable region of Ozarow & Leung
      - Capacity not known – an outer bound
Note ...

- Feedback is instantaneous & noiseless
  - May not be practical
  - Noiseless feedback: Satellite links
  - Upper bound on practical feedback
Single user results

- Discrete memoryless channel
  - Capacity not changed by feedback: Shannon
  - Same result for white Gaussian channel

- Gaussian channel with correlated noise
  - Capacity is at most doubled by feedback
    - Proof by Pinsker and Ebert
  - Capacity is at most increased by half a bit
    - Proof by Cover and Pombra
MAC without feedback

- Capacity region:

\[ R_1 \cdot I(X_1; Y|X_2) \]
\[ R_2 \cdot I(X_2; Y|X_1) \]
\[ R_1 + R_2 \cdot I(X_1; X_2; Y) \]

Where \( p(x_1; x_2; y) = p(y|x_1; x_2)p(x_1)p(x_2) \)

Any point in the convex hull can be achieved by time sharing
Achievable region of Cover & Leung

\[ R_1 < I(X_1; Y_jX_2; U) \]
\[ R_2 < I(X_2; Y_jX_1; U) \]
\[ R_1 + R_2 < I(X_1; X_2; Y) \]

Where the cardinality of U is bounded as:

\[ jjUjj = \min \{ jjX_1jj \mid jjX_2jj; jjYjj \} \]

And: \[ p(u; x_1; x_2; y) = p(u)p(x_1|u)p(x_2|u)p(y|x_1; x_2) \]

Proof: Block Markov Coding
Block Markov Coding

☐ Tx B blocks of length n each

☐ Tx:
  ■ At block b superimpose & send
    ☐ New data
    ☐ An index to help Rx decode block b-1
  ■ Decode other Tx message by feedback

☐ Rx:
  ■ Decodes block b-1 using index of block b
  ■ Limits the possible estimates for block b
Block Markov Coding 2

\[ w_1 2 f 1 \phi \phi 2^{n R_1} g \quad w_2 2 f 1 \phi \phi 2^{n R_2} g \quad j 2 f 1 \phi \phi J g \]

Choose \( u^n(j) : Q^n_{i=1} p(u_i) \). This codebook is the same for both Tx.

Tx1: Choose \( x_1^n(w_1; j) : Q^n_{i=1} p(x_1 i j u_i(j)) \). Send \( x_1^n(w_1; j) \) at block b.

Tx2: Choose \( x_2^n(w_2; j) : Q^n_{i=1} p(x_2 i j u_i(j)) \). Send \( x_2^n(w_2; j) \) at block b.

Rx: Find unique \( j \) s.t. \( (u^n(j); y^n) \in A^n_2 \)

Find all pairs \( (w_1; w_2) \) s.t. \( (x_1^n(w_1; j); x_2^n(w_2; j); y^n) \in A^n_2 \)

Tx1: Find unique \( w_2 \) s.t. \( (x_1^n(w_1; j); x_2^n(w_2; j); y^n) \in A^n_2 \)

Tx2: Find unique \( w_1 \) s.t. \( (x_1^n(w_1; j); x_2^n(w_2; j); y^n) \in A^n_2 \)
An Outer Bound ...

\[ R_1 \cdot I(X_1; Y | X_2) \]
\[ R_2 \cdot I(X_2; Y | X_1) \]
\[ R_1 + R_2 \cdot I(X_1; X_2; Y) \]

Let \( C_0 \) denote the convex hull over all rate pairs \((R_1; R_2)\) satisfying the above equations for a given \( p(x_1; x_2) \).

\[ C_{fb} \mu C_0 \]
White Gaussian MAC: no feedback

For two Tx with average power constraints:

\[ E[X_1^2] \cdot P_1; \quad E[X_2^2] \cdot P_2 \]

\[ R_1 \cdot \frac{1}{2} \log(1 + \frac{P_1}{N}) \]

\[ R_2 \cdot \frac{1}{2} \log(1 + \frac{P_2}{N}) \]

\[ R_1 + R_2 \cdot \frac{1}{2} \log(1 + \frac{P_1 + P_2}{N}) \]
White Gaussian MAC with feedback

For 2 Tx capacity found by Ozarow:

\[
R_1 \cdot \frac{1}{2} \log(1 + \frac{P_1}{N} (1 - \frac{1}{2}))
\]

\[
R_2 \cdot \frac{1}{2} \log(1 + \frac{P_2}{N} (1 - \frac{1}{2}))
\]

\[
R_1 + R_2 \cdot \frac{1}{2} \log(1 + \frac{P_1 + P_2 + 2\frac{P_1 P_2}{N}}{1})
\]

Capacity region is the convex hull of all rate pairs \((R_1; R_2)\)
Satisfying above equations for a \(\frac{1}{2} : 0 \cdot \frac{1}{2} : 1\).
Deterministic Coding with feedback

- Proposed by Schalkwijk & Kailath
  - Map messages to real numbers in [-.5, .5]
  - Send that number instead of the message
  - Rx feeds back an estimate of that number
  - After receiving the feedback
    - Tx sends the error in Rx estimate
    - Rx updates its estimate at each iteration
    - Rx feeds back its new estimate
    - Repeat until error variance is small enough
    - Closest value to Rx final estimate is chosen
Deterministic Coding with feedback 2

Tx $i$ has message $m_i \in \{0; \ldots; M_i - 1\}$ to transmit for $i = 1; 2$.

**Tx $i$:** $\mu_i = \frac{m_i}{M_i - 1} \cdot \frac{1}{2}$ \quad $\mu_i : U[0.5; 0.5]$  

At time $k = 1$: Tx $1$ sends $\frac{p}{12}P_1\mu_1$ while Tx $2$ is quiet.  
At time $k = 2$: Tx $2$ sends $\frac{p}{12}P_2\mu_2$ while Tx $1$ is quiet.

**Rx:** At time $k = 1$ receives $Y_1 = \frac{p}{12}P_1\mu_1 + Z_1$  
And forms the estimate $\hat{\mu}_1 = \hat{\mu}_2 = \frac{pY_1}{12P_1}$  

At time $k = 2$ receives $Y_2 = \frac{p}{12}P_2\mu_2 + Z_2$ and forms $\hat{\mu}_2 = \frac{pY_2}{12P_2}$.
Deterministic Coding with feedback

Rx feeds back its estimates to the transmitters

At time $k + 1$ where $k \leq 2$:

$\text{Tx 1: } X_{1;k+1} = q \frac{p_{1;k+1}}{\tilde{\mu}_{1;k+1}} \tilde{\mu}_{1,k} = \hat{\mu}_{1,k} i \mu_{1}$

$\text{Tx 2: } X_{2;k+1} = q \frac{p_{2;k+1}}{\tilde{\mu}_{2;k+1}} \text{sign}(\frac{1}{k}) \tilde{\mu}_{2,k} = \hat{\mu}_{2,k} i \mu_{2}$

Rx: Forms the LMMSE

$\hat{\mu}_{i,k+1} = \hat{\mu}_{i,k} i \frac{Y_{k+1;i;k}}{Y_{k+1;i;k}^2} Y_{k+1}$
MAC: Feedback Vs. No feedback

2 Tx + 1 Rx

P_1 = P_2 = 10

N=1
Broadcast Channel

- Degraded BC without feedback

\[ R_1 \cdot I(X; Y_j U) \]
\[ R_2 \cdot I(U; Z) \]

Where \( j \) is the minimum of \( j \) given \( j \) for \( X, Y, Z \).

Capacity is the convex hull over all achievable rate pairs for a distribution \( p(u; x; y; z) = p(u)p(x|u)p(y; z|x) \).
Degraded BC with feedback

- Physically degraded BC
  - Capacity does not increase with feedback
  - Converse similar to BC without feedback
  - Proved by El Gamal
- Physically degraded white Gaussian BC
  - Converse proved by El Gamal
  - Uses the modified entropy power inequality

\[
\frac{2H(Y)}{n} e^{n} - e^{2H(Z)} + e^{\sum_{i=1}^{P} H(X_i | Y_{i-1})}
\]
Degraded BC with feedback 2

- Degraded white Gaussian BC
  - Feedback strictly enlarges capacity
  - Achievable region of Ozarow & Leung
    - Proof: Deterministic coding similar to MAC
  - Capacity not known
    - An outer bound by Ozarow & Leung
White Gaussian BC: Model

\[ z_1 : N(0; N) \]

\[ z_2 : N(0; N_1) \]

\[ z_3 : N(0; N_3) \]

\[ w_1; w_2 \]

\[ w_1 \]

\[ w_2 \]
White Gaussian BC: no feedback

\[ R_1 = \frac{1}{2} \log \left( 1 + \frac{\circ P}{N + N_1} \right) \]

\[ R_2 = \frac{1}{2} \log \left( 1 + \frac{(1 \circ P)}{N + N_2 + \circ P} \right) \]

\[ \circ 2 [0; 1] \text{ allocates total power between } R_1 \text{ and } R_2 \]
Ozarow & Leung Region

\[ R_1 = \frac{1}{2} \log \frac{N + N_1 + P}{N + N_2 + P \frac{D^n}{2} g^2 (1 + \frac{1}{2})} \]

\[ R_2 = \frac{1}{2} \log \frac{N + N_2 + P}{N + N_2 + P \frac{D^n}{2} (1 + \frac{1}{2})} \]

Where \( D^n = 1 + g^2 + 2g^{1/2} \) and \( \frac{1}{2} \) is given as a function of \( N; N_1; N_2; P; g \)

Ex. \( P = 10; N = 0; N_1 = N_2 = 1 \)

No feedback: \( R_1 = R_2 = 0.59947 \) nats
Feedback: \( R_1 = R_2 = 0.70468 \) nats
Outer bound: \( R_1 = R_2 = 0.71956 \) nats
Outer bound for BC with feedback

- Let Rx1 know the output of Rx2
- Results in a physically degraded BC
  - Capacity with feedback is known

\[
R_1 = \frac{1}{2} \log \left( 1 + \frac{\mathbb{R}P}{N + \frac{N_1 N_2}{N_1 + N_2}} \right)
\]
\[
R_2 = \frac{1}{2} \log \left( 1 + \frac{(1 - \mathbb{R})P}{N + N_2 + \mathbb{R}P} \right)
\]

Reverse the order of \(R_1; R_2\) to get another outer bound

Capacity region is included in the intersection of these two regions
Inner-Outer bounds for BC

\[ P = 10 \]
\[ N = 0 \]
\[ N_1 = N_2 = 1 \]
Conclusions

- **MAC**
  - Feedback strictly enlarges the capacity
  - Only case where capacity is known:
    - White Gaussian MAC with 2 Tx

- **BC**
  - Physically degraded channels:
    - No increase in capacity
  - White Gaussian BC
    - Feedback strictly enlarges capacity
Thank You!