Linear Coding for Fading Channels

Jinjun Xiao

EE 8510 Project Presentation

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Problem Formulation

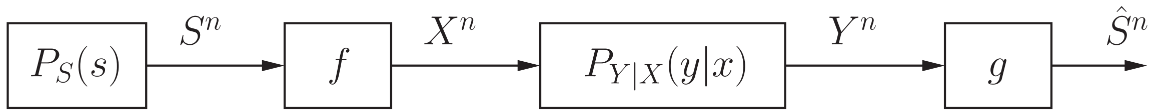
- Consider a **memoryless Gaussian source** $\{s(i) : i \in \mathbb{Z}^+\}$ transmitted through a **discrete memoryless fading channel**

$$y(i) = h(i)x(i) + w(i),$$

  - $w(i)$ are AWGN with unitary variance
  - $h(i)$ are i.i.d. fading with known distribution h

- There is **channel state information (CSI)** at receiver only.

- A **source-channel coding system** is illustrated as



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  - the distortion measure is mean squared distortion

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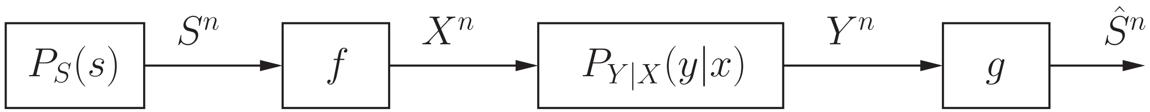


  - there is average power constraint P on X^n
  - the distortion measure is mean squared distortion

- Although optimal performance can be achieved by **separate source and channel coding**, it is worthwhile to consider joint source/channel coding with low complexity and short delay, such as **linear coding**.

Linear Coding of Block Length $n = 1$

- A **single-letter linear coding** $(n = 1)$:

$$S(i) \longrightarrow \boxed{f(i)} \longrightarrow X(i) \longrightarrow \boxed{\text{channel}} \longrightarrow Y(i) \longrightarrow \boxed{g(i)} \longrightarrow \hat{S}(i),$$

in which

$$X(i) = \sqrt{\frac{P}{\sigma_s^2}} S(i); \quad \hat{S}(i) = \frac{Ph^2}{Ph^2 + 1} y(i).$$

The achieved mean squared distortion

$$D_u(P) = \sigma_S^2 E_h \left\{ \frac{1}{1 + h^2 P} \right\}.$$

- When h **is deterministic** (the channel is **AWGN**), the **single-letter linear coding** is optimal.

- How about the case if h **is random**? What is the performance of **linear coding** (of block length $n = 1$, or when block length increases)?

Linear Coding is Optimal Among All Single-letter Codes

- A **single-letter coding system**:

$$S(i) \longrightarrow \boxed{f(i)} \longrightarrow X(i) \longrightarrow \boxed{\text{channel}} \longrightarrow Y(i) \longrightarrow \boxed{g(i)} \longrightarrow \hat{S}(i)$$

- **Lemma:** Let S **be a Gaussian random variable** with variance σ_S^2, and $\hat{S}$ be any random variable jointly distributed with S. Then

$$\frac{E(|S-\hat{S}|^2)}{\sigma_S^2} \geq \exp\left(-2I(S;\hat{S})\right).$$

- By **data processing inequality**, we obtain that for any single-letter coding $\{f(i), g(i)\}$ when there is power constraint $P(i)$, and the fading coefficient is $h(i)$, then the achieved distortion at time i:

$$E\left(|S(i)-\hat{S}(i)|^2 \,\middle|\, h(i)\right) \geq \frac{\sigma_S^2}{1+h^2(i)P(i)}.$$

Linear Coding is Optimal Among All Single-letter Codes

- Therefore **the average distortion for letter** $S(i)$

$$E(|S(i)-\hat{S}(i)|^2) = E_{h(i)}\left\{E\left(|S(i)-\hat{S}(i)|^2 \middle| h(i)\right)\right\} \geq \sigma_S^2 E_{h(i)}\left\{\frac{1}{1+h(i)^2P(i)}\right\},$$

where equality is obtained by linear coding.

- Finally, **uniform power allocation is optimal** due to the convex property of

$$D(P(i)) \stackrel{\text{def}}{=} \sigma_S^2 E_h\left\{\frac{1}{1+h^2P(i)}\right\}$$

- Therefore **linear coding with uniform power allocation is optimal among all single-letter codes**.

- **Is linear coding optimal in Shannon's sense?**

Condition for Linear Coding Achieving Shannon's Bound

- The **rate-distortion function** and **channel capacity** are

$$R(D) = \frac{1}{2}\log^{+}\frac{\sigma_S^2}{D}, \qquad C(P) = E_h\left\{\frac{1}{2}\log(1+h^2P)\right\}.$$

Combining the above two formulas, we obtain **the Shannon's bound**

$$D_c(P) = \sigma_S^2 \exp\left(E_h\left\{\log\frac{1}{1+h^2P}\right\}\right).$$

- The **linear coding with block length** $n = 1$ has average distortion

$$D_u(P) = \sigma_S^2 E_h\left\{\frac{1}{1+h^2P}\right\}.$$

- $D_u(P) \geq D_c(P)$ from concavity of the log-function. The equality holds iff $\frac{1}{1+h^2P} = const.$

- **Linear coding** (with block length $n = 1$) is optimal in Shannon's sense iff $|h|$ **is deterministic**.
  - If h is real, $h \equiv \pm c$.
  - If h is complex, then h should be distributed on a circle.

Linear Coding of Finite Block Length

- We consider a **linear coding with block length** n. The encoder is given by a $n \times n$ matrix F, and the decoder is a MMSE decoder.

$$S^{(n)} \longrightarrow \boxed{F} \longrightarrow X^{(n)} \longrightarrow \boxed{\text{channel}} \longrightarrow Y^{(n)} \longrightarrow \boxed{\text{MMSE}} \longrightarrow \hat{S}^{(n)}$$

- Under such a set-up, the **achieved MMSE** is

$$D(H;F) = \frac{1}{n}\,\mathrm{tr}\left((HF\Omega_S F^T H^T + I)^{-1}\Omega_S\right).$$

The **power constraint** implies

$$P(F) = \mathrm{tr}(F\Omega_S F^T) \le nP.$$

- Thus, we can solve the following problem for **optimal** F

$$\begin{aligned} \min \quad & E_H\left\{\mathrm{tr}\left((HF\Omega_S F^T H^T + I)^{-1}\Omega_S\right)\right\} \\ \text{s.t.} \quad & \mathrm{tr}(F\Omega_S F^T) \le nP. \end{aligned}$$

Linear Coding of Finite Block Length

- When channel is **DMC** and source is **memoryless**, we have

$$H = \text{diag}(h(1), \ldots, h(n)), \quad \Omega_S = \sigma_S^2 I$$

- Introducing $Q = FF^T \succeq 0$, the problem is changed to

$$\begin{aligned} \min \quad & E_H \left\{ \text{tr}(HQH^T + \sigma_S^{-2} I)^{-1} \right\} \\ \text{s.t.} \quad & \text{tr}(Q) \leq nP/\sigma_S^2, \quad Q \succeq 0. \end{aligned}$$

- **Lemma:** For any $R \succ 0$, $\text{tr}(R^{-1}) \geq \sum_{i=1}^{n} R_{ii}^{-1}$, and equality holds iff R is diagonal.

- Optimal solution gives **diagonal** $Q^* = FF^T$. Thus, any $F^* = \sqrt{Q^*}U$ where U is unitary is an optimal solution. Specifically, if we take $U = I$, we can obtain a **diagonal** F^*.

- **Any linear coding can be achieved in a single-letter form without performance loss.**

A Lower Bound on the Performance of Linear Coding

- Introducing $h_0^2 = E(|h^2|)$, then we obtain

$$D_c(P) = \sigma_S^2 \exp\left(E_h\left\{\log \frac{1}{1+h^2P}\right\}\right) \geq \sigma_S^2 \frac{1}{1+h_0^2P}.$$

- The **linear coding** achieves distortion

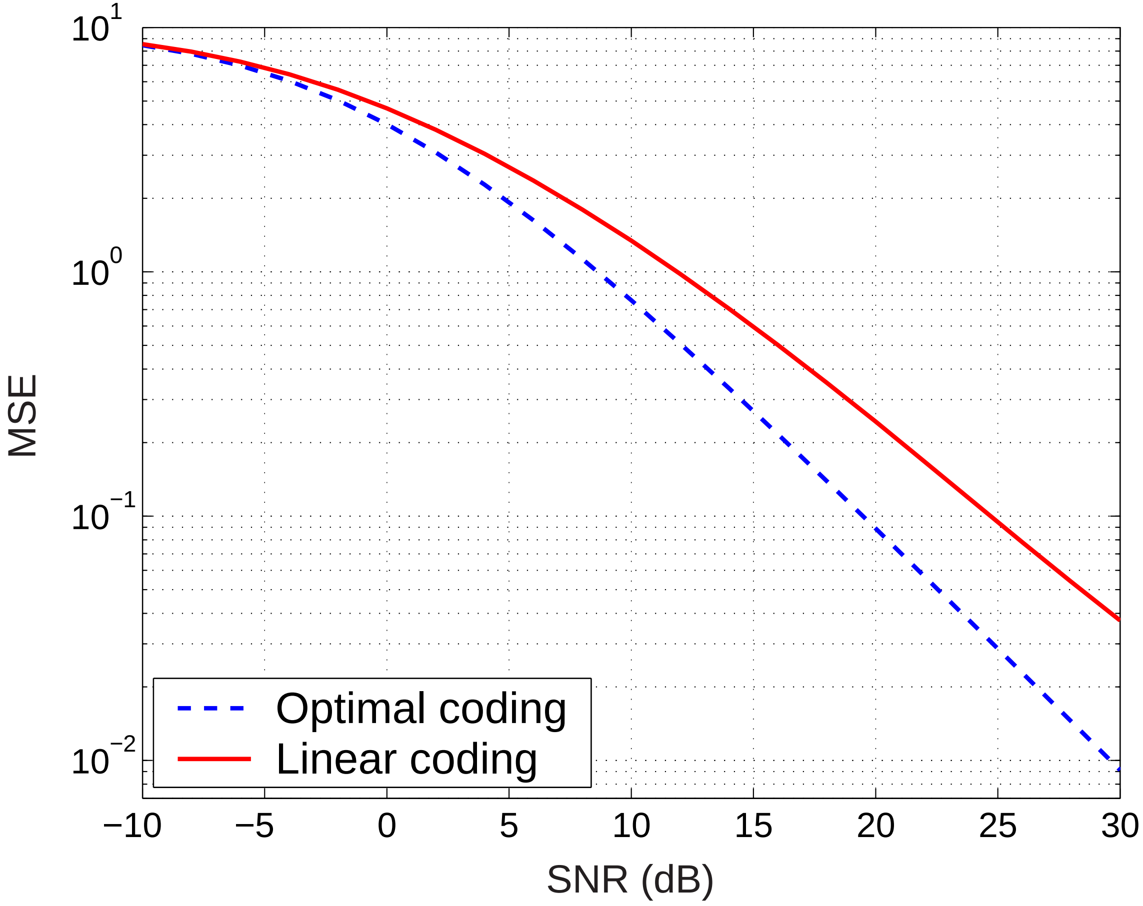
$$D_u(P) = E_h(D(h)) = \sigma_S^2 E_h\left\{\frac{1}{1+h^2P}\right\}.$$

- We can verify that

$$0 \leq \frac{D_u(P) - D_c(P)}{D_c(P)} \leq E_h\left\{\frac{(h^2 - h_0^2)P}{1+h^2P}\right\} \leq P\sqrt{\mathrm{Var}(|h|^2)}.$$

- **Linear coding** is close to optimal in Shannon's sense if
  - $\mathrm{Var}(|h|^2)$ is small, or
  - If P is small such as applications in sensor networks.

Simulations



- Rayleigh fading with $P = 1$; source $\sigma_S^2 = 10$

Linear Coding When There is TX CSI

- **It still hods that**
  - every linear coding is equivalent to a linear coding of block length $n = 1$;
  - linear coding is optimal among all single-letter codes.

- The **optimal power loading** can be solved from

$$\begin{aligned} \min \quad & \sigma_S^2 E_h \left\{ \frac{1}{1 + h^2 P(h)} \right\} \\ \text{s.t.} \quad & E_h\{P(h)\} = P, \quad P(h) \geq 0. \end{aligned}$$

The optimal power loading (in terms of fading state h) can be solved analytically,

$$P^{opt}(h) = \frac{1}{|h|} \left(u_0 - \frac{1}{|h|} \right)^+, \quad \text{for some } u_0 > 0.$$

- **Performance loss** compared to the optimal coding can also be lower bounded in terms of the statistic of $|h|$ and power constraint P.

Concluding Remarks

Considered **a memoryless Gaussian source** transmitted through **a DMC fading channel with AWGN**:

- Among all single-letter codes, linear coding is optimal;
- Every linear coding is equivalent to a linear coding of block length $n = 1$;
- Linear coding in general can not approach Shannon's bound;
- The performance loss of linear coding compared to the optimal coding can be lower bounded in terms of $\text{Var}(|h|^2)$ and power constraint P.

Thanks!