Linear Coding for Fading Channels

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Problem Formulation

- Consider a **memoryless Gaussian source** \( \{ s(i) : i \in \mathbb{Z}^+ \} \) transmitted through a **discrete memoryless fading channel**

\[
y(i) = h(i)x(i) + w(i),
\]

- \( w(i) \) are **AWGN** with unitary variance
- \( h(i) \) are i.i.d. fading with known distribution \( h \)

- There is **channel state information (CSI)** at receiver only.

- A **source-channel coding system** is illustrated as

\[
\begin{array}{cccc}
P_S(s) & S^n & f & X^n \\
& & P_{Y|X}(y|x) & Y^n \\
& & g & \hat{S}^n
\end{array}
\]

- there is average **power constraint** \( P \) on \( X^n \)
- the distortion measure is **mean squared distortion**
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\end{array}
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- the distortion measure is **mean squared distortion**

- Although optimal performance can be achieved by **separate source and channel coding**, it is worthwhile to consider joint source/channel coding with **low complexity and short delay**, such as **linear coding**.
A single-letter linear coding \((n = 1)\):

\[
S(i) \rightarrow f(i) \rightarrow X(i) \rightarrow \text{channel} \rightarrow Y(i) \rightarrow g(i) \rightarrow \hat{S}(i),
\]

in which

\[
X(i) = \sqrt{\frac{P}{\sigma_s^2}} S(i); \quad \hat{S}(i) = \frac{P h^2}{P h^2 + 1} y(i).
\]

The achieved mean squared distortion

\[
D_u(P) = \sigma_s^2 E_h \left\{ \frac{1}{1 + h^2 P} \right\}.
\]

When \(h\) is deterministic (the channel is AWGN), the single-letter linear coding is optimal.

How about the case if \(h\) is random? What is the performance of linear coding (of block length \(n = 1\), or when block length increases)?
Linear Coding is Optimal Among All Single-letter Codes

- A single-letter coding system:

  \[ S(i) \rightarrow f(i) \rightarrow X(i) \rightarrow \text{channel} \rightarrow Y(i) \rightarrow g(i) \rightarrow \hat{S}(i) \]

- Lemma: Let \( S \) be a Gaussian random variable with variance \( \sigma^2_S \), and \( \hat{S} \) be any random variable jointly distributed with \( S \). Then

  \[
  \frac{E(|S - \hat{S}|^2)}{\sigma^2_S} \geq \exp \left( -2I(S; \hat{S}) \right). 
  \]

- By data processing inequality, we obtain that for any single-letter coding \( \{f(i), g(i)\} \) when there is power constraint \( P(i) \), and the fading coefficient is \( h(i) \), then the achieved distortion at time \( i \):

  \[
  E \left( |S(i) - \hat{S}(i)|^2 \middle| h(i) \right) \geq \frac{\sigma^2_S}{1 + h^2(i)P(i)}.
  \]
Linear Coding is Optimal Among All Single-letter Codes

- Therefore the average distortion for letter $S(i)$

$$E(|S(i) - \hat{S}(i)|^2) = E_{h(i)} \left\{ E \left( |S(i) - \hat{S}(i)|^2 \mid h(i) \right) \right\} \geq \sigma^2_S E_{h(i)} \left\{ \frac{1}{1 + h(i)^2 P(i)} \right\},$$

where equality is obtained by linear coding.

- Finally, uniform power allocation is optimal due to the convex property of

$$D(P(i)) \overset{\text{def}}{=} \sigma^2_S E_h \left\{ \frac{1}{1 + h^2 P(i)} \right\}$$

- Therefore linear coding with uniform power allocation is optimal among all single-letter codes.

- Is linear coding optimal in Shannon’s sense?
Condition for Linear Coding Achieving Shannon’s Bound

- The rate-distortion function and channel capacity are

\[ R(D) = \frac{1}{2} \log + \frac{\sigma_S^2}{D}, \quad C(P) = E_h \left\{ \frac{1}{2} \log (1 + h^2 P) \right\}. \]

Combining the above two formulas, we obtain the Shannon’s bound

\[ D_c(P) = \sigma_S^2 \exp \left( E_h \left\{ \log \frac{1}{1 + h^2 P} \right\} \right). \]

- The linear coding with block length \( n = 1 \) has average distortion

\[ D_u(P) = \sigma_S^2 E_h \left\{ \frac{1}{1 + h^2 P} \right\}. \]

- \( D_u(P) \geq D_c(P) \) from concavity of the log-function. The equality holds iff \( \frac{1}{1 + h^2 P} = \text{const}. \)

- Linear coding (with block length \( n = 1 \)) is optimal in Shannon’s sense iff \( |h| \) is deterministic.
  - If \( h \) is real, \( h \equiv \pm c \).
  - If \( h \) is complex, then \( h \) should be distributed on a circle.
**Linear Coding of Finite Block Length**

- We consider a **linear coding with block length** \( n \). The encoder is given by a \( n \times n \) matrix \( F \), and the decoder is a MMSE decoder.

\[
S^{(n)} \rightarrow F \rightarrow X^{(n)} \rightarrow \text{channel} \rightarrow Y^{(n)} \rightarrow \text{MMSE} \rightarrow \hat{S}^{(n)}
\]

- Under such a set-up, the **achieved MMSE** is

\[
D(H; F) = \frac{1}{n} \text{tr} \left( (HF\Omega_S F^T H^T + I)^{-1} \Omega_S \right).
\]

The **power constraint** implies

\[
P(F) = \text{tr}(F\Omega_S F^T) \leq nP.
\]

- Thus, we can solve the following problem for **optimal** \( F \)

\[
\min_{EH} \left\{ \text{tr} \left( (HF\Omega_S F^T H^T + I)^{-1} \Omega_S \right) \right\}
\]

\[
\text{s.t.} \quad \text{tr}(F\Omega_S F^T) \leq nP.
\]
Linear Coding of Finite Block Length

- When channel is DMC and source is memoryless, we have
  \[ H = \text{diag}(h(1), \ldots, h(n)), \quad \Omega_S = \sigma_S^2 I \]

- Introducing \( Q = FF^T \succeq 0 \), the problem is changed to
  \[
  \min_{E_H} \left\{ \text{tr}(HQH^T + \sigma_S^{-2} I)^{-1} \right\} \\
  \text{s.t.} \quad \text{tr}(Q) \leq nP/\sigma_S^2, \quad Q \succeq 0.
  \]

- **Lemma:** For any \( R \succ 0 \), \( \text{tr}(R^{-1}) \geq \sum_{i=1}^{n} R_{ii}^{-1} \), and equality holds iff \( R \) is diagonal.

- Optimal solution gives **diagonal** \( Q^* = FF^T \). Thus, any \( F^* = \sqrt{Q^*}U \) where \( U \) is unitary is an optimal solution. Specifically, if we take \( U = I \), we can obtain a **diagonal** \( F^* \).

- Any linear coding can be achieved in a single-letter form without performance loss.
A Lower Bound on the Performance of Linear Coding

- Introducing $h_0^2 = E(|h^2|)$, then we obtain

$$D_c(P) = \sigma_s^2 \exp \left( E_h \left\{ \log \frac{1}{1 + h^2 P} \right\} \right) \geq \sigma_s^2 \frac{1}{1 + h_0^2 P}.$$  

- The linear coding achieves distortion

$$D_u(P) = E_h(D(h)) = \sigma_s^2 E_h \left\{ \frac{1}{1 + h^2 P} \right\}.$$  

- We can verify that

$$0 \leq \frac{D_u(P) - D_c(P)}{D_c(P)} \leq E_h \left\{ \frac{(h^2 - h_0^2)P}{1 + h^2 P} \right\} \leq P \sqrt{\text{Var}(|h|^2)}.$$  

- **Linear coding** is close to optimal in Shannon’s sense if
  - $\text{Var}(|h|^2)$ is small, or
  - If $P$ is small such as applications in sensor networks.
• Rayleigh fading with $P = 1$; source $\sigma_S^2 = 10$
Linear Coding When There is TX CSI

- It still holds that
  - every linear coding is equivalent to a linear coding of block length $n = 1$;
  - linear coding is optimal among all single-letter codes.

- The **optimal power loading** can be solved from

\[
\min \sigma_S^2 E_h \left\{ \frac{1}{1 + h^2 P(h)} \right\}
\]
\[
s.t. \ E_h \{P(h)\} = P, \ P(h) \geq 0.
\]

The optimal power loading (in terms of fading state $h$) can be solved analytically,

\[
P^{opt}(h) = \frac{1}{|h|} \left( u_0 - \frac{1}{|h|} \right)^+, \quad \text{for some } u_0 > 0.
\]

- **Performance loss** compared to the optimal coding can also be lower bounded in terms of the statistic of $|h|$ and power constraint $P$. 
Concluding Remarks

Considered a memoryless Gaussian source transmitted through a DMC fading channel with AWGN:

- Among all single-letter codes, linear coding is optimal;
- Every linear coding is equivalent to a linear coding of block length $n = 1$;
- Linear coding in general can not approach Shannon’s bound;
- The performance loss of linear coding compared to the optimal coding can be lower bounded in terms of $\text{Var}(|h|^2)$ and power constraint $P$. 
Thanks!