Final Exam

You have 2 hours to complete this exam. You must show your work to receive credit.

1. **Basics** (15 pts)
   Let $X_1$ and $X_2$ be i.i.d. Bern(1/2) random variables, and let $Y = \max(X_1, X_2)$. Compute: (5 pts each)
   
   (a) $H(Y)$
   (b) $I(X_1; Y)$
   (c) $I(X_1, X_2; Y)$

2. **Source Coding** (20 pts)
   (a) Which of the following codes are optimal prefix-free codes for the given source distribution? Briefly justify each answer. (10 pts total)

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<tr>
<th>$x$</th>
<th>$p(x)$</th>
<th>$C(x)$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>110</td>
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<tr>
<td>2</td>
<td>0.5</td>
<td>0</td>
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<td>3</td>
<td>0.1</td>
<td>10</td>
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<td>4</td>
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   (b) Consider the following source $X$:

   $X = \begin{cases} 
   1, & \text{with probability } 0.25 \\
   2, & \text{with probability } 0.25 \\
   3, & \text{with probability } p \\
   4, & \text{with probability } (0.5 - p) 
   \end{cases}$

   for $0 < p < 0.5$. For what values of $p$ does the optimal prefix-free code have expected length equal to 2? (10 pts)
3. **Channel Capacity** (25 pts)
Compute the capacity of the following channels:

(a) $C = ?$ (5 pts)

(b) $C = ?$ (10 pts)

(c) $C = ?$ (10 pts)
4. **Differential Entropy** (15 pts)

Let \( X \) be a continuous random variable with support \( S = [-1, 1] \) (i.e., \( f(x) > 0 \) for \(-1 \leq x \leq 1\) and \( f(x) = 0 \) for \( x < -1 \) and \( x > 1 \)). Assume \( h(X) \) is finite. Define the random variable \( Y \) as:

\[
Y = \begin{cases}
+ a & \text{with probability } 1/2 \\
- a & \text{with probability } 1/2
\end{cases}
\]

for some constant \( a \geq 0 \). Assume \( X \) and \( Y \) are independent. Let \( Z = X + Y \).

(a) Compute \( h(Z) \) in terms of \( h(X) \) for \( a > 1 \). (10 pts)

(Hint: The quantity \( h(Z) \) is finite.)

(b) Does the same answer hold if \( a < 1 \)? Why or why not? (5 pts)

5. **Rate Distortion** (25 pts)

Consider a ternary source and reconstruction alphabet \( (\mathcal{X} = \{0, 1, 2\}, \hat{\mathcal{X}} = \{0, 1, 2\}) \). Assume the source has a uniform distribution, i.e. \( p(X = 0) = p(X = 1) = p(X = 2) = 1/3 \), and let the distortion measure be given by the following matrix:

\[
d(x, \hat{x}) = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
\]

or equivalently

\[
d(x, \hat{x}) = \begin{cases}
1 & \text{if } (x = 0, \hat{x} = 2) \text{ or } (x = 1, \hat{x} = 0) \text{ or } (x = 2, \hat{x} = 1) \\
0 & \text{otherwise.}
\end{cases}
\]

(a) Compute an expression for the expected distortion \( E[d(x, \hat{x})] \). (5 pts)

(b) Compute the rate distortion function at \( D = 1/3 \), i.e., \( R(D = 1/3) \). (10 pts)

(c) Compute the rate distortion function at \( D = 0 \), i.e., \( R(D = 0) \). (10 pts)

(Hint: \( R(0) \) is strictly smaller than \( H(X) \) because there are two zero-distortion reconstructions for each source symbol.)