OrnsteinUhlenbeck Process

→ The Orstein Uhlenbeck process is governed by the equation

$$V(t+dt) - V(t) = -\gamma V(t)\,dt + \sqrt{\delta^2 dt}\, N_t^{t+dt}(0,1)$$

where $N_t^{t+dt}(0,1)$ has the same properties as obtained for the Wiener process derivation earlier namely that

$N_{t_1}^{t_2}(0,1)$ is independent of $N_{t_3}^{t_4}(0,1)$

if $(t_1, t_2) \cap (t_3, t_4) = \phi$.

Assume that $V(0)$ is a Gaussian random Variable

Note that

$$V(dt) = V(0) - \gamma V(0)dt + \sqrt{\delta^2 dt}\; N_0^{dt}(0,1)$$

$$= (1-\gamma)\; V(0)\; dt + N_0^{dt}(0, \delta^2 dt)$$

$$V(2dt) = (1-\gamma)\, V(dt)\,dt + N_{dt}^{2dt}(0, \delta^2 dt)$$

Solution

$$\Rightarrow \quad V(2dt) = (1-r)\left[V(0)(1-r) + N_0^{dt}(0, \delta^2 dt)\right] + N_{dt}^{2d}(0, \delta^2 dt)$$

which is a linear combination of

$V(0)$, $N_0^{dt}(0, \delta^2 t)$ and $N_{dt}^{2dt}(0, \delta^2 t)$

Evidently, as $V(0)$, $N_0^{dt}(0, \delta^2 t)$ and $N_{dt}^{2dt}(0, \delta^2 t)$

are all Gaussian and independent

$V(2dt)$ is also Gaussian (Normal).

In a similar fashion it can be shown that

$V(ndt)$ is a linear combination of $V(0)$,

$N_0^{dt}(0,1)$, $N_{dt}^{2dt}(0,1)$, ..., $N_{(n-1)dt}^{ndt}(0,1)$

which are independent Gaussian.

Therefore, it follows that

$V(t) = V(ndt)$ is a Gaussian (Normal)
random variable that is completely characterized

Solution

by its mean and variance.

Thus,

$$V(t) \sim N\{\langle V(t)\rangle, \mathrm{var}\, V(t)\}$$

Therefore the pdf of $V(t)$ can be ascertained by determining its mean and variance.

Now,

$$V(t+dt) - V(t) = -\gamma V(t)dt + \sqrt{\delta^2 dt}\, N_t^{t+dt}(0,1)$$

Taking the mean on both sides we have

$$\langle V(t+dt)\rangle - \langle V(t)\rangle = -\gamma\langle V(t)\rangle dt$$

$$\Rightarrow \quad \frac{\langle V(t+dt)\rangle - \langle V(t)\rangle}{dt} = -\gamma\langle V(t)\rangle$$

Let $\mu(t) \doteq \langle V(t)\rangle$; a deterministic variable

$$\therefore \quad \frac{\mu(t+dt) - \mu(t)}{dt} = -\gamma\mu(t)$$

Solution (mean)

$$\Rightarrow \lim_{dt \to 0} \frac{\mu(t+dt) - \mu(t)}{dt} = -r\mu(t)$$

$$\Rightarrow \frac{d\mu(t)}{dt} = -r\mu(t).$$

which has a solution

$$\mu(t) = \mu(0)\, e^{-rt}$$

$$\therefore \quad \boxed{\langle v(t) \rangle = \langle v(0) \rangle e^{-rt}}$$

Now we obtain the variance.

Let $\alpha(t) \doteq \langle v(t)^2 \rangle$.

a deterministic function of time.

Then

$$\frac{d\alpha(t)}{dt} = \lim_{dt \to 0} \frac{\alpha(t+dt) - \alpha(t)}{dt}$$

$$\alpha(t+dt) = \langle v(t+dt)^2 \rangle$$

$$= \left\langle \left[(1-\gamma dt) v(t) + N_t^{t+dt}(0, \delta^2 dt) \right]^2 \right\rangle$$

$$= \Big\langle (1-\gamma dt)^2 v^2(t) + 2(1-\gamma dt)\, v(t) \left[N_t^{t+dt}(0, \delta^2 dt) \right] + \left[N_t^{t+dt}(0, \delta^2 dt) \right]^2 \Big\rangle$$

$$= (1-\gamma dt)^2 \langle v^2(t) \rangle + 2(1-\gamma dt) \langle v(t)\, N_t^{t+dt}(0, \delta^2 dt) \rangle + \langle \left[N_t^{t+dt}(0, \delta^2 dt) \right]^2 \rangle$$

$$= (1-\gamma dt)^2 \alpha(t) + 0 + \delta^2 dt$$

where we have used the fact that

Variance

$$\langle V(t)\, N_t^{t+dt}(0,1)\rangle = 0$$

as $V(t)$ is a linear combination

of $\{N_0^{dt}(0,1),\ N_{dt}^{2dt},\ \dots\ N_{t-dt}^{t}(0,1)\}$

all of which are independent of

$N_t^{t+dt}(0,1)$

$$\Big[\ V(t) = a_0 N_0^{dt}(0,1) + a_1 N_{dt}^{2dt}(0,1) + \dots + a_n N_{t-dt}^{t}(0,1)$$

and $\langle V(t) N_t^{t+dt}(0,1)\rangle = a_0 \langle N_0^{dt}(0,1)\, N_t^{t+dt}(0,1)\rangle + \dots + a_n \langle N_{t-dt}^{t}(0,1)\, N_t^{t+dt}(0,1)\rangle$

$$= 0\Big].$$

$$\therefore\ \alpha(t+dt) = (1-\gamma dt)^2 \alpha(t) + \delta^2 dt$$

$$= (1 + \gamma^2 dt^2 - 2\gamma dt)\alpha(t) + \delta^2 dt$$

$$= \alpha(t) + \gamma^2 dt^2 - 2\gamma dt\, \alpha(t) + \delta^2 dt$$

$$\Rightarrow \quad \frac{\alpha(t+dt) - \alpha(t)}{dt} = \gamma^2 dt - 2\gamma\alpha(t) + \delta^2$$

Variance

$$\lim_{dt\to 0} \frac{\alpha(t+dt)-\alpha(t)}{dt} = \lim_{dt\to 0} \gamma^2 dt - 2\gamma\alpha(t) + \delta^2$$

$$\therefore \quad \frac{d\alpha(t)}{dt} = -2\gamma\alpha(t) + \delta^2 .$$

Which has a solution

$$\alpha(t) = e^{-2\gamma t}\alpha(0) + \int_0^t e^{-2\gamma(t-\tau)}\delta^2 \, d\tau$$

$$= e^{-2\gamma t}\alpha(0) + e^{-2\gamma t}\delta^2 \int_0^t e^{2\gamma\tau} d\tau$$

$$= e^{-2\gamma t}\alpha(0) + \delta^2 e^{-\gamma t}\left[\frac{1}{2\gamma} e^{2\gamma\tau}\right]_0^t$$

$$= e^{-2\gamma t}\alpha(0) + \frac{\delta^2}{2\gamma} e^{-\gamma t}\left[e^{-2\gamma t} - 1\right]$$

$$= e^{-2\gamma t}\alpha(0) + \frac{\delta^2}{2\gamma}\left[1 - e^{-2\gamma t}\right]$$

$$\therefore \langle V(t)^2\rangle = \langle V^2(0)\rangle e^{-2\gamma t} + \frac{\delta^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)$$

$$\mathrm{Var}\{V(t)\} = \langle V^2(t)\rangle - \langle V(t)\rangle^2$$

$$= \langle V^2(0)\rangle e^{-2\gamma t} + \frac{\delta^2}{2\gamma}\left(1 - e^{-2\gamma t}\right) - \langle V(0)\rangle^2 e^{-2\gamma t}$$

Variance

$$\Rightarrow \mathrm{Var}\{v(t)\} = [\langle v^2(0)\rangle - \langle v(0)\rangle^2]\, e^{-2\gamma t} + \frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right).$$

$$= \mathrm{Var}\{v(0)\}\, e^{-2\gamma t} + \frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)$$

If $v(0)$ is deterministic with value η_0 then $\mathrm{Var}\{v(0)\} = 0$ and

$$\mathrm{Var}\{v(t)\} = \frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)$$

Thus, $\langle v(t)\rangle = \eta_0 e^{-\gamma t}$

and $\mathrm{Var}\{v(t)\} = \frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)$ $\Bigg\}$ if $v(0) = \eta_0$ a constant

in which case as $v(t)$ is Normal

$$v(t) \sim N_0^t\left[\eta_0 e^{-\gamma t}, \frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)\right]$$

$$p(\eta, t) = \frac{1}{\sqrt{2\pi \frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)}} \exp\left[-\frac{\left(\eta-\eta_0 e^{-\gamma t}\right)^2}{2\frac{\delta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)}\right]$$

Generalization

Suppose the only information on $V(0)$ is that $V(0)$ is independent of $N_{t_1}^{t_2}(0,1)$.

Then

$$V(t+dt) - V(t) = -r V(t) dt + \sqrt{\delta^2 dt}\; N_t^{t+dt}(0,1)$$

$$V(t+dt) = (1 - r dt) V(t) + \sqrt{\delta^2 dt}\; N_t^{t+dt}(0,1)$$

$$\Rightarrow V(dt) = (1 - r dt) V(0) + \sqrt{\delta^2 dt}\; N_0^{dt}(0,1).$$

$$\Rightarrow V(2dt) = (1 - r dt) V(dt) + \sqrt{\delta^2 dt}\; N_{dt}^{2dt}(0,1)$$

$$\Rightarrow \quad = (1 - r dt)\left[(1 - r dt) V(0) + \sqrt{\delta^2 dt}\; N_0^{dt}(0,1)\right] + \sqrt{\delta^2 dt}\; N_{dt}^{2dt}(0,1)$$

$$= (1 - r dt)^2 V(0) + (1 - r dt)\sqrt{\delta^2 dt}\; N_0^{dt}(0,1) + \sqrt{\delta^2 dt}\; N_{dt}^{2dt}(0,1)$$

$$V(3dt) = (1 - r dt) V(2dt) + N_{2dt}^{3dt}(0, \delta^2 dt)$$

$$= (1 - r dt)^3 V(0) + (1 - r dt)^2 \sqrt{\delta^2 dt}\; N_0^{dt}(0,1) + (1 - r dt) N_{dt}(0, \delta^2 t) + N_{2dt}^{3dt}(0, \delta^2 dt)$$

$$\therefore \quad V(ndt) = (1 - r dt)^n V(0) + Z(ndt)$$

where $Z(ndt)$ is the part of $V(ndt)$ excluding contribution from $V(0)$.

Initial Condition

$V(ndt) = (1-rdt)^n V(0) + Z(ndt)$

or letting $ndt = t$ and letting $dt \to 0$ $(n \to \infty)$

we have

$$V(t) = \lim_{n\to\infty} \left(1 - r\frac{t}{n}\right)^n V(0) + Z(t)$$

$Z(t)$ we have already shown is

$$N_0^t\left(0, \frac{\delta^2}{2r}\left(1-e^{-rt}\right)\right)$$

also

$$\lim_{n\to\infty} \left(1-\frac{rt}{n}\right)^n = e^{-rt}$$

$$\therefore V(t) = e^{-rt} V(0) + N_0^t\left[0, \frac{\delta^2}{2r}\left(1-e^{-rt}\right)\right]$$

Drift

Consider the equation

$$V(t+dt) - V(t) = -\gamma\,[V(t) - V_d]\,dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)$$

Let

$$Z(t) := V(t) - V_d \quad \text{then}$$

$$Z(t+dt) - Z(t) = V(t+dt) - V(t)$$

$$= -\gamma\,[V(t) - V_d]\,dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)$$

$$\Rightarrow \quad Z(t+dt) - Z(t) = -\gamma Z(t)\,dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)$$

which is the equation we have solved earlier with the solution

$$Z(t) = e^{-\gamma t} Z(0) + N_0^t\left(0, \frac{\beta^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)\right)$$

$$\therefore \quad V(t) - V_d = e^{-\gamma t}[V(0) - V_d] + N_0^t\left(0, \frac{\beta^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)\right)$$

$$\Rightarrow \quad V(t) = V_d + e^{-\gamma t} V(0) - e^{-\gamma t} V_d + N_0^t\left(0, \frac{\beta^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)\right)$$

Drift

$$V(t) = V_d + e^{-rt} V(0) - e^{-rt} V_d + N_0^t\left[0, \frac{\beta^2}{2r}(1-e^{-rt})\right]$$

$$= e^{-rt} V(0) + N_0^t\left[V_d(1-e^{-rt}), \frac{\beta^2}{2r}(1-e^{-rt})\right]$$

In particular if $V(0) = \langle V(0)\rangle$ a constant then

$$\boxed{V(t) = N_0^t\left[V_d + e^{-rt}(\langle V(0)\rangle - V_d), \frac{\beta^2}{2r}(1-e^{-rt})\right]}$$

We have seen that

$$V(t) = N_0^t \left(v_d + e^{-\gamma t}(v_0 - v_d), \frac{\beta^2}{2\gamma}(1 - e^{-2\gamma t}) \right)$$

and therefore

$$\lim_{t \to \infty} \operatorname{Var}\{V(t)\} = \frac{\beta^2}{2\gamma} \quad (\text{with } v_d = 0)$$

In the limit $t \to \infty$, the Brownian particle also approaches thermal equilibrium and therefore each degree of freedom is associated with $\frac{k_B T}{2}$ of energy and thus

$$\frac{1}{2} M \langle V^2(\infty) \rangle = \frac{1}{2} k_B T$$

$$\Rightarrow \quad \frac{M \beta^2}{2\gamma} = k_B T$$

usually Stokes law holds $\Rightarrow \gamma = \frac{6\pi \eta r}{M}$

where η is the liquid viscosity and r is the radius, k_B is the Boltzman Constant

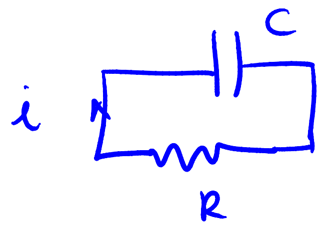
Thus,

$$\beta^2 = \frac{2\gamma k_B T}{M}$$

$$= \frac{2(6\pi\eta r)k_B T}{M^2}$$

$$\boxed{\beta = \frac{1}{M}\sqrt{12\pi\eta r k_B T}}$$

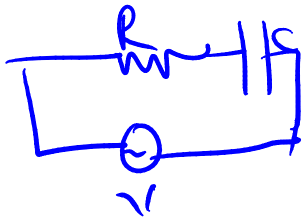
Consider an R-C circuit



where the ambient temperature is T degrees kelvin.

Suppose the initial charge on the capacitor is q_0. As charge is transferred from the capacitor to the resistor element, charged particles transfer energy to atoms of the resistor. However, the atoms of the resistor (as the resistor is not at absolute zero) contain thermal energy and vibrate randomly due to this energy. Thus, the

Johnson-noise



resistor atoms impart energy to the charge carriers.

Thus,

$$iR + \frac{Q}{C} - (\text{Johnson noise}) = 0$$

$$\Rightarrow \frac{dQ}{dt}R + \frac{Q}{C} - (\text{Johnson noise}) = 0$$

$$\Rightarrow dQ = -\frac{Q}{RC}dt + \frac{(\text{Johnson noise})}{R}dt$$

Johnson Noise can be modeled by assuming

$$\sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1) = \frac{\text{"Johnson noise"}}{R}dt$$

$$\Rightarrow dQ = -\frac{Q}{RC}dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)$$

$$\Rightarrow Q(t) = N_0^t\left[q_0 e^{-\gamma t}, \frac{\beta^2}{2\gamma}\left(1-e^{-2\gamma t}\right)\right]$$

where $\gamma = \frac{1}{RC}$

Indeed

$$\lim_{t\to\infty} \mathrm{Var}\{Q(t)\} = \frac{\beta^2}{2\gamma}$$

with $\lim_{t\to\infty} \langle Q(t) \rangle = 0$

Thus, the energy stored in the capacitor in thermal equilibrium ($t \to \infty$) is

$$\left\langle \frac{1}{2} C V^2 \right\rangle = \left\langle \frac{1}{2} C \frac{Q^2}{C^2} \right\rangle = \left\langle \frac{1}{2C} Q^2 \right\rangle = \frac{1}{2C} \langle Q^2 \rangle$$

$$= \frac{1}{2C} \frac{\beta^2}{2\gamma}$$

$$\therefore \frac{\beta^2}{4C\gamma} = \frac{1}{2} k_B T$$

$$\Rightarrow \beta^2 = 2C\gamma k_B T = \frac{2\cancel{C} k_B T}{R\cancel{C}} = \frac{2 k_B T}{R}$$

$$\Rightarrow \beta = \sqrt{\frac{2 k_B T}{R}}.$$

$$\Rightarrow Q(t) = N_0^t \left[q_0 e^{-t/RC},\ kTC\left(1 - e^{-2t/RC}\right) \right]$$

We have shown that if

$$V(t+dt) - V(t) = -\gamma V(t)dt + v_d\, dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1) \quad \text{---- (VE)}$$

Then

$$V(t) = N_0^t \left[v_0 e^{-\gamma t} + v_d(1-e^{-\gamma t}),\ \frac{\beta^2}{2\gamma}(1-e^{-2\gamma t}) \right]$$

where v_0 is the initial condition.

The above equation can be considered as the velocity update equation.

Now we derive the statistics of the position that results from the velocity update equation as given by (VE).

Integrating the O-U process.

Note that

$$v(t+dt) - v(t) = -\gamma v(t)dt + \gamma_g dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)$$

implies that

$$v(t) \in \text{Span}\{N_0^{dt}(0,1),\ N_{dt}^{2dt}(0,1), \dots, N_{t-dt}^{dt}(0,1)\}$$

Now,

$$x(t+dt) - x(t) = v(t)dt$$

and therefore

$$x(dt) = x(0) + v(0)dt$$
$$= x_0 + v_0 dt$$

$$x(2dt) = x(dt) + v(dt)dt$$

$$= x_0 + v_0 dt + v(dt)dt$$

$$\therefore\ x(2dt) \in \text{Span}\{N_0^{dt}(0,1)\}$$

$$x(3dt) = x(2dt) + v(2dt)dt$$
$$\in \text{Span}\{N_0^{dt}(0,1), N_{dt}^{2dt}(0,1)\}$$

and therefore it can be concluded that

$$x(ndt) \in \text{Span}\{N_0^{dt}(0,1) \dots N_{(n-2)dt}^{(n-1)dt}(0,1)\}$$

$$\therefore\ x(t) \in \text{Span}\{N_0^{dt}(0,1), \dots, N_{t-2dt}^{t-dt}(0,1)\}$$

Integration of O-U process.

Thus, it can be concluded that $V(t)$ and $x(t)$ are linear combinations of elements of the same set of independent Normals. Thus, $X(t)$ and $V(t)$ are jointly Gaussian and therefore, the joint pdf of $X(t)$ and $V(t)$ is completely determined by $\langle X(t)\rangle$, $\langle V(t)\rangle$, $Var\{X(t)\}$

$Var\langle X(t)\rangle$ and $Cov\{X(t), V(t)\}$

We have already determined $\langle V(t)\rangle$ and $Var\{V(t)\}$. Now we derive $\langle X(t)\rangle$.

Note that

$$x(t+dt) - x(t) = v(t)\,dt$$

$$\Rightarrow \langle x(t+dt) - x(t)\rangle = \langle v(t)\rangle dt$$

$$\Rightarrow \langle x(t+dt)\rangle - \langle x(t)\rangle = \langle v(t)\rangle dt$$

Let $\theta(t) := \langle x(t)\rangle$ a deterministic quantity

$$\therefore \quad \theta(t+dt) - \theta(t) = [v_0 e^{-\gamma t} + v_d(1-e^{-\gamma t})]dt$$

$$\Rightarrow \lim_{dt\to 0} \frac{\theta(t+dt) - \theta(t)}{dt} = v_0 e^{-\gamma t} + v_d(1-e^{-\gamma t})$$

$$\Rightarrow \frac{d\theta(t)}{dt} = v_0 e^{-\gamma t} + v_d(1-e^{-\gamma t})$$

$$\Rightarrow \theta(t) = \theta(0) + \int_0^t v_0 e^{-\gamma t} dt + \int_0^t v_d(1-e^{-\gamma t})dt$$

$$= x_0 + \frac{v_0}{-\gamma} e^{-\gamma t}\Big|_0^t + v_d\left(t - \frac{e^{-\gamma t}}{-\gamma}\right)\Big|_0^t$$

$$\Rightarrow \boxed{\langle x(t)\rangle = x_0 + \frac{v_0}{\gamma}[1 - e^{-\gamma t}] + \frac{v_d}{\gamma}[\gamma t + e^{-\gamma t} - 1]}$$

Variance of the Langevin Process

For convenience of notation we will denote by

$$dx := x(t+dt) - x(t) = v\,dt$$

$$dv := v(t+dt) - v(t)$$

and thus, $x(t+dt) = x(t) + dx$
$= x + dx$

$$v(t+dt) = v + dv.$$

$$\langle x^2(t+dt)\rangle - \langle x^2(t)\rangle = \langle x+dx\rangle^2 - \langle x^2\rangle$$

$$= \langle x^2 + 2x\,dx + dx^2\rangle - \langle x^2\rangle$$

$$= \langle x^2\rangle + \langle 2x\,dx\rangle + \langle dx^2\rangle - \langle x^2\rangle$$

$$= \langle 2x\,dx\rangle + \langle dx^2\rangle$$

$$= \langle 2x\,dx\rangle + \langle v^2 dt^2\rangle$$

Variance of the Langevin Process

Thus

$$\langle x(t+dt)^2\rangle - \langle x^2(t)\rangle = \langle 2x\,dx\rangle + \langle v^2\rangle dt^2$$

$$= \langle 2x\,v\,dt\rangle + \langle v^2\rangle dt^2$$

$$= 2\langle xv\rangle dt + \langle v^2\rangle dt^2$$

Let $\quad n(t) \doteq \langle x^2(t)\rangle - \langle x\rangle^2 = \langle x^2(t)\rangle - \theta^2(t)$

a deterministic quantity

$$\therefore \quad n(t+dt) - n(t) = 2\langle xv\rangle dt + \langle v^2\rangle dt^2 - \left[\theta^2(t+dt) - \theta^2(t)\right]$$

$$\lim_{dt\to 0} \frac{n(t+dt) - n(t)}{dt} = 2\langle xv\rangle - \frac{d\,\theta^2(t)}{dt}$$

$$\frac{dn(t)}{dt} = 2\langle xv\rangle - 2\theta(t)\frac{d\theta(t)}{dt}$$

$$= 2\langle xv\rangle - 2\langle x\rangle\langle v\rangle$$

$$= 2\,\mathrm{Cov}[x, v]$$

Let

$m(t) := \mathrm{Cov}\{x(t)\, v(t)\}$

$= \langle x(t) v(t) \rangle - \langle x(t) \rangle \langle v(t) \rangle$

$= \langle x v \rangle - \langle x \rangle \langle v \rangle$

$m(t+dt) = \langle x(t+dt)\, v(t+dt) \rangle - \langle x(t+dt) \rangle \langle v(t+dt) \rangle$

$= \langle (x+dx)(v+dv) \rangle - \langle x+dx \rangle \langle v+dv \rangle$

$= \langle xv + x\,dv + v\,dx + dx\,dv \rangle$

$- (\langle x \rangle + \langle dx \rangle)(\langle v \rangle + \langle dv \rangle)$

$= \langle xv \rangle + \langle x\,dv \rangle + \langle v\,dx \rangle + \langle dx\,dv \rangle$

$- (\langle x \rangle + d\langle x \rangle)(\langle v \rangle + d\langle v \rangle)$

$= \langle xv \rangle + \langle x\,dv \rangle + \langle v\,dx \rangle + \langle dx\,dv \rangle - [\langle x \rangle \langle v \rangle + \langle x \rangle d\langle v \rangle + \langle v \rangle d\langle x \rangle + d\langle x \rangle d\langle v \rangle]$

$\therefore \quad m(t+dt) - m(t) = \langle x\,dv \rangle + \langle v\,dx \rangle + \langle dx\,dv \rangle - \langle x \rangle d\langle v \rangle - \langle v \rangle d\langle x \rangle - d\langle x \rangle d\langle v \rangle$

$m(t+dt) - m(t) = \langle x\,dv\rangle + \langle v\,dx\rangle + \langle dx\,dv\rangle$
$- \langle x\rangle d\langle v\rangle - \langle v\rangle d\langle x\rangle - d\langle x\rangle d\langle v\rangle$

Note that $dx = v\,dt \Rightarrow \langle dx\rangle = d\langle x\rangle = \langle v\rangle dt$

$dv = -\gamma v\,dt + \gamma v_d\,dt + \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)$

As $\langle dx\,dv\rangle = \langle v\,dv\rangle dt$

Note that

$\langle v\,dv\rangle = \langle -\gamma v^2 dt + \gamma v_d v\,dt + v\sqrt{\beta^2 dt}\, N_t(0,1)\rangle$

$= -\gamma\langle v^2\rangle dt + \gamma v_d \langle v\rangle dt + \langle v(t)\, N_t^{t+dt}(0,1)\rangle \sqrt{\beta^2}\,dt$

Note that $\langle v(t)\, N_t^{t+dt}(0,1)\rangle = 0$
as $v(t) \in \mathrm{Span}\{N_0^{dt}(0,1), \dots N_{t-dt}^{\cancel{dt}}(0,1)\}$

$\therefore \langle v\,dv\rangle = -\langle v^2\rangle dt + \gamma v_d \langle v\rangle dt$ and
$\langle v\,dv\rangle dt = (-\langle v^2\rangle + \gamma v_d \langle v\rangle) dt^2$

Also

$d\langle x\rangle d\langle v\rangle = \langle v\rangle d\langle v\rangle dt$

and therefore

Note that $\langle v\rangle\, d\langle v\rangle$

$$= \langle v\rangle \langle dv\rangle$$

$$= \langle v\rangle \left[-\gamma\langle v\rangle dt + \gamma v_d\, dt + \sqrt{\beta^2 dt}\, \langle N_t^{t+dt}(0,1)\rangle \right]$$

$$= \langle v\rangle \left[-\gamma\langle v\rangle + \gamma v_d \right] dt$$

$$\therefore\ \langle v\rangle\, d\langle v\rangle\, dt = \langle v\rangle \left[-\gamma\langle v\rangle + \gamma v_d \right] dt^2$$

Thus

$$m(t+dt) - m(t) = \langle x\, dv\rangle + \langle v\, dx\rangle + \langle dx\, dv\rangle - \langle x\rangle d\langle v\rangle - \langle v\rangle d\langle x\rangle - d\langle x\rangle d\langle v\rangle$$

$$= \langle x\, dv\rangle + \langle v\, dx\rangle - \langle x\rangle d\langle v\rangle - \langle v\rangle d\langle x\rangle + \langle dx\, dv\rangle - d\langle x\rangle d\langle v\rangle$$

and

$$\lim_{dt\to 0} \left(m(t+dt) - m(t) \right) = \lim_{dt\to 0} \left(\langle x\, dv\rangle + \langle v\, dx\rangle - \langle x\rangle d\langle v\rangle - \langle v\rangle d\langle x\rangle \right) \Big/ dt$$

as $\displaystyle \lim_{dt\to 0} \frac{\langle dx\, dv\rangle - d\langle x\rangle d\langle v\rangle}{dt} = 0$

$\therefore$ We need to evaluate

$$\frac{\langle x\, dv\rangle + \langle v\, dx\rangle - \langle x\rangle d\langle v\rangle - \langle v\rangle d\langle x\rangle}{dt}$$

$$= \frac{1}{dt}\Big[\langle x\,(-\gamma v\,dt + v_d\gamma\,dt + \sqrt{\beta^2 dt}\,N_t^{t+dt}(0,1))\rangle$$
$$+ \langle v\, v\rangle dt - \langle x\rangle(-\gamma\langle v\rangle dt + v_d\gamma\,dt)$$
$$- \langle v\rangle\langle v\rangle dt\Big]$$

$$= \frac{1}{dt}\Big[-\gamma\langle xv\rangle dt + \gamma v_d\,\cancel{dt\langle x\rangle dt} + \sqrt{\beta^2 dt}\,\langle x N_t^{t+dt}(0,1)\rangle$$
$$+ \langle v^2\rangle dt + \gamma\langle x\rangle\langle v\rangle dt - \langle x\rangle\gamma \cancel{v_d}\, dt\Big]$$
$$- \langle v\rangle^2 dt$$

$$= \frac{1}{dt}\Big[-\gamma\langle xv\rangle dt + 0 + \langle v^2\rangle dt + \gamma\langle x\rangle\langle v\rangle dt\Big]$$
$$- \langle v\rangle^2 dt$$

$$= -\gamma\big[\langle x v\rangle - \langle x\rangle\langle v\rangle\big] + \langle v^2\rangle - \langle v\rangle^2$$

where $\langle x N_t^{t+dt}(0,1)\rangle = \langle x\rangle \underbrace{\langle N_t^{t+dt}(0,1)\rangle}_{=0}$

as $x = x(t) \in \text{Span}\{N_0^{dt} \dots N_{t-2dt}^{t-dt}(0,1)\}$

$\therefore \quad \dfrac{dm(t)}{dt} = -\gamma\, m(t) + \mathrm{Var}\{x(t)\}.$

$$= -\gamma m(t) + \frac{\beta^2}{2\gamma}\left(1-e^{-2\gamma t}\right).$$

which has a solution

$$m(t) = e^{-\gamma t} m(0) + \int_0^t e^{-\gamma(t-z)} \frac{\beta^2}{2\gamma}\left(1-e^{-2\gamma z}\right) dz$$

$$= e^{-\gamma t} m(0) + \frac{\beta^2}{2\gamma} e^{-\gamma t} \int_0^t \left(1-e^{-2\gamma z}\right) e^{\gamma z} dz$$

$$= e^{-\gamma t} m(0) + \frac{\beta^2}{2\gamma} e^{-\gamma t} \int_0^t \left(e^{\gamma z} - e^{-\gamma z}\right) dz$$

$$m(t) = \frac{\beta^2}{2\gamma^2}\left(1 - 2e^{-\gamma t} + e^{-2\gamma t}\right)$$

Also, we have

$$\frac{d\,\mathrm{Var}\{x(t)\}}{dt} = 2m(t) = \frac{\beta^2}{2\gamma^2}\left(1 - e^{-\gamma t} + e^{-2\gamma t}\right)$$

$$\therefore \quad Var\{x(t)\} = \frac{\beta^2}{\gamma^2}\left[t - \frac{2}{\gamma}(1-e^{-\gamma t}) + \frac{1}{2\gamma}(1-e^{-2\gamma t})\right]$$

$$\langle x(t)\rangle = x_0 + \frac{v_0}{\gamma}(1-e^{-\gamma t}) + \frac{v_d}{\gamma}(\gamma t + e^{-\gamma t} - 1).$$