Week 3:

Note Title

9/21/2009

①

• We have seen that many systems admit a mathematical description of the form

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du.$$

with initial conditions $x(t_0)$, with initial time t_0.

• We have also seen that the solution to the vector differential equation is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

which is called the "Variation of parameters formula"

① e^{At} is defined by the following

$$e^{x} = \left[I + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right]$$

Some properties of e^{x}

① $e^{x} = \left[I + x + \frac{x^2}{2!} + \cdots \right]$

so

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \cdots$$

$$\Rightarrow \frac{d(e^{At})}{dt} = 0 + A + \frac{2A^2 t}{2!} + \frac{3A^3 t^2}{3!} + \cdots$$

$$= A + A^2 t + \frac{A^3 t^2}{2!} + \cdots$$

$$= \left[I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots \right] A$$

$$= e^{At} A.$$

$\therefore$ $$\boxed{\frac{d(e^{At})}{dt} = e^{At} A}$$

② Suppose $A \in R^{n \times n}$
$B \in R^{n \times n}$

and $AB = BA$

then

$$e^{A+B} = e^A e^B = e^B e^A.$$

Ineed

$$e^{A+B} = \left(I + (A+B) + \frac{(A+B)^2}{2} + \cdots \right)$$

$$e^A e^B = \left[I + A + \frac{A^2}{2!} + \cdots \right] \left[I + B + \frac{B^2}{2!} + \cdots \right]$$

$$= \Big[I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots$$

$$+ A + AB + \frac{AB^2}{2!} + \cdots$$

$$+ \frac{A^2}{2!} + \frac{A^2 B}{2!} + \cdots \Big]$$

$$= \left[I + (A+B) + \left(\frac{A^2}{2} + AB + \frac{B^2}{2} \right) + \cdots \right]$$

$$= \left[I + (A+B) + \frac{1}{2}(A+B)^2 + \cdots \right]$$

$$= e^{A+B}.$$

$$\therefore \boxed{e^{A+B} = e^A e^B = e^B e^A \quad \text{if } AB = BA}$$

(c) $(e^{x})^{-1} = e^{-x}$

Indeed, this follows easily as

$-x^{2} = x(-x) = (-x)x$

$\therefore \quad e^{x+(-x)} = e^{0} = I$

as $(x)(-x) = (-x)x$

$e^{x+(-x)} = e^{x}e^{-x}$

$\therefore \quad e^{x}e^{-x} = I$

$\Rightarrow \quad e^{-x} = (e^{x})^{-1}$

e^{At} for Some Simpler Cases

(a) $e^{\Lambda t}$ when Λ is diagonal

Let Λ be a diagonal square matrix with

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

then

$$e^{\Lambda t} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & & 0 \\ & \lambda_1 t & & \\ & & \ddots & \\ 0 & & & \lambda_n t \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\lambda_1^2 t^2}{2!} & & & 0 \\ & \frac{\lambda_2^2 t^2}{2!} & & \\ & & \ddots & \\ 0 & & & \frac{\lambda_n^2 t}{2!} \end{bmatrix} + \cdots$$

$\therefore$

$$e^{At} = \begin{bmatrix} 1+\lambda_1 t+\frac{(\lambda_1 t)^2}{2!}+\cdots & & & \\ & 1+\lambda_2 t+\frac{(\lambda_2 t)^3}{2!} & & \\ & & \ddots & \\ & & & 1+(\lambda_n t)+\frac{(\lambda_n t)^2}{2!}+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

(b) e^{At} when A is "diagonalizable"

Suppose
$$A = P \Lambda P^{-1}$$
where Λ is diagonal given by
$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

then note that
$$A^2 = (P \Lambda P^{-1})(P \Lambda P^{-1})$$
$$= P \Lambda (P^{-1} P) \Lambda P^{-1}$$
$$= P \Lambda I \Lambda P^{-1}$$
$$= P \Lambda^2 P^{-1}$$
$$A^3 = A^2 \cdot A$$
$$= (P \Lambda^2 P^{-1})(P \Lambda P^{-1})$$
$$= P \Lambda^2 (P^{-1} P) \Lambda P^{-1}$$
$$= P \Lambda^3 P^{-1}$$
and in general
$$A^m = P \Lambda^m P^{-1}$$

$\therefore$

$$e^{At} = \left[I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots \right]$$

$$= \left[PP^{-1} + P\Lambda P^{-1} t + \frac{P\Lambda^2 P^{-1} t^2}{2!} + \frac{P\Lambda^3 P^{-1} t^3}{3!} + \cdots \right]$$

$$= P\left[P^{-1} + \Lambda P^{-1} t + \frac{\Lambda^2 P^{-1} t^2}{2!} + \frac{\Lambda^3 P^{-1} t^3}{3!} + \cdots \right]$$

$$= P\left[I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \frac{\Lambda^3 t^3}{3!} + \cdots \right] P^{-1}$$

$$= P e^{\Lambda t} P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix} P^{-1}$$

with $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$.

Fact: If A has distinct eigenvalues then there exists a matrix P such that

$$A = P \Lambda P^{-1}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$ where λ_i are the eigenvalues of the matrix A.

Examples:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

then $e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$

Note that if $x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ then

$$e^{At}\bar{x}(0) = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} \\ 0 \\ 0 \end{bmatrix}$$

and $\|e^{At}x(0)\|_2 \to \infty$ as $t \to \infty$.

$\therefore$ Then the initial condition response of $\dot{x} = Ax$ goes to ∞ for some $x(0)$.

Example:

Suppose

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

then

$$e^{At}x(0) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}x_1(0) \\ e^{-2t}x_2(0) \\ e^{-3t}x_3(0) \end{bmatrix}$$

and $e^{At}x(0) \longrightarrow 0$ as $t \to \infty$ for any $x(0)$.

Clearly if $A = P \Lambda P^{-1}$

then $e^{At} = P e^{\Lambda t} P^{-1}$ with $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

and $Re(\lambda_i) < 0$

$\therefore \quad e^{At} x(0) = P e^{\Lambda t} P^{-1} x(0)$

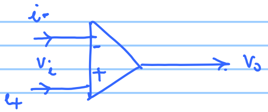
$= P e^{\Lambda t} y(0)$ where $y(0) = P^{-1} x(0)$

and as $e^{\Lambda t} y(0) \to 0$ as $t \to \infty$

$e^{At} x(0) \to 0$ as $t \to \infty$.

How to "Realize" a System

Op Amp:

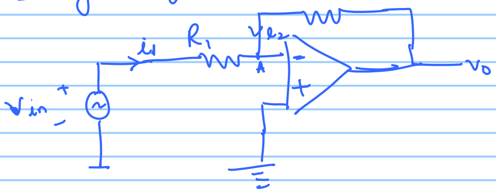


Assumptions

(1) $v_i = 0$

(2) $i_- = i_+ = 0$.

Scaling a Signal:



$V_+ = V_- = 0$

$\therefore \quad V_A = 0$.

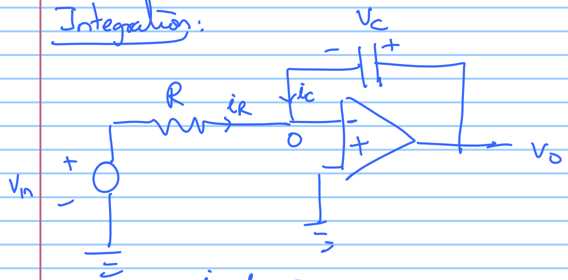
$$\Rightarrow \frac{V_0 - V_A}{R_2} + \frac{V_{in} - V_A}{R_1} = 0$$

$$\Rightarrow \frac{V_0}{R_2} = \frac{-V_{in}}{R_1}$$

$$\Rightarrow \boxed{\frac{V_0}{V_{in}} = -\frac{R_2}{R_1}}$$

$\therefore$ V_0 is $-\frac{R_2}{R_1}$ scaled version of V_{in}

Integration:



$$i_c + i_R = 0$$

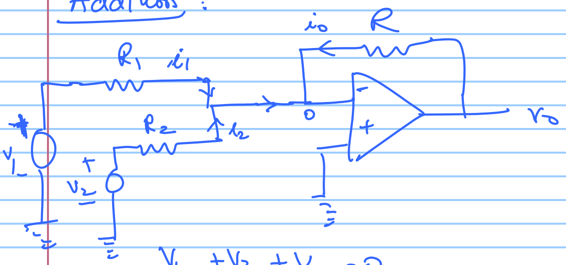
$$\Rightarrow C\frac{dV_0}{dt} + \frac{V_{in}}{R} = 0$$

$$\Rightarrow \frac{dV_0}{dt} = -\frac{V_{in}}{RC}$$

$$\Rightarrow \quad V_o(t) - V_o(t_0) = -\frac{1}{RC}\int_{t_0}^{t} V_{in}(\tau)\, d\tau.$$

$$\Rightarrow \quad \boxed{V_o(t) = V_o(t_0) - \frac{1}{RC}\int_{t_0}^{t} V_{in}(\tau)\, d\tau}$$

Addition:



$$\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_o}{R} = 0$$

$$\Rightarrow \quad V_o = -\left(\frac{R}{R_1}V_1 + \frac{R}{R_2}V_2\right)$$

$$= -(\alpha_1 V_1 + \alpha_2 V_2).$$

Thus, using op-amps signals can be added, scaled and integrated.

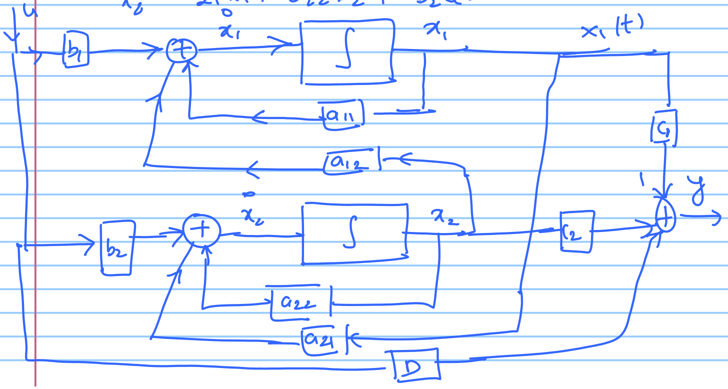
Now consider a State-Space realization.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\therefore \quad \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1 u$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2 u.$$



Example:

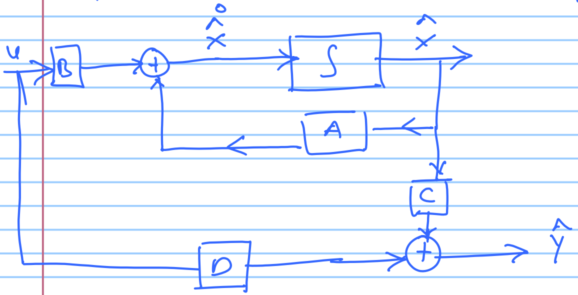
Given a physical system with a description

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$x(0)$ being the initial condition

One can realize a "circuit" that implements the above model as follows



i.e.

$$\dot{\hat{x}} = A\hat{x} + Bu$$

$$y = C\hat{x} + Du.$$

$\hat{x}(0)$ being the initial condition.

Now consider the error in the estimate $\hat{x}$ and x. with

$$e \triangleq x - \hat{x}$$

$$\Rightarrow \frac{de}{dt} = \frac{dx}{dt} - \frac{d\hat{x}}{dt}$$

$$= (Ax + Bu) - (A\hat{x} + Bu)$$

$$= A(x - \hat{x})$$

$$= Ae$$

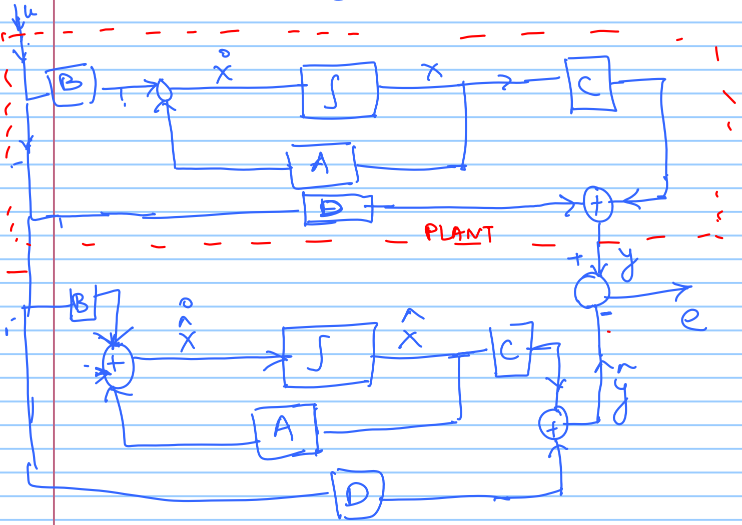
with $e(0) = x(0) - \hat{x}(0)$

Therefore

$$e(t) = [\exp(At)]\, e(0)$$

and $e(t) \to 0$ as $t \to \infty$ if the original system $\dot{x} = Ax + Bu$ is asymptotically stable. Thus, the "tracking error" $e = x - \hat{x}$ goes to zero if A is stable which implies the circuit will provide a faithful estimate of the state of the system.

This Scheme is shown below:

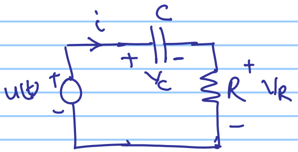


Systems

* A system processes a "time" signal to yield another "time" signal as output

Example:

Consider



$$i = C\frac{dV_c}{dt} \quad ; \quad V_R = iR$$

Also, $\quad V_c + V_R - u = 0$

$$\Rightarrow \quad V_c + iR - u = 0$$

$$\Rightarrow \quad CR\frac{dV_c}{dt} + V_c = u(t)$$

$$\Rightarrow \quad \frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{u(t)}{RC}$$

Assume initial capacitor voltage is zero

i.e. $\quad V_c(0) = 0$

Therefore the physics is described by

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{u(t)}{V_c} \; ; \quad V_c(0) = 0$$

where $u(t)$ is the input to the "System".

(*) Suppose current through the resistor is the signal of interest. Let the output be denoted by y. Thus

$$y(t) = i(t) = C \frac{dV_c(t)}{dt}$$

(*) Thus the input $u(t)$ gives the output according to the following rules

input

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{u(t)}{RC} \; ; \quad V_c(0) = 0$$

output $\longrightarrow y(t) = \dfrac{C\, dV_c(t)}{dt}$

(*) Note that if the above rules are followed there is no "ambiguity" of what the signal

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$y(t)$ is going to be given $u(t)$ is specified.

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The Impulse Response and Convolution:

* Note that the impulse function has the property that

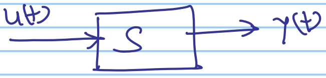
$$\int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, d\tau = f(t)$$

which is called the "Sampling" property

Linearity of a System:

Suppose S is a system with input $u(t)$ and output $y(t)$ denoted by

$$u(t) = (Sy)(t).$$



S is said to linear if

$$S(u_1+u_2) = (Su_1) + (Su_2)$$

and

$$S(\alpha u) = \alpha\, S(u) \quad \text{where } \alpha \text{ is a constant}$$

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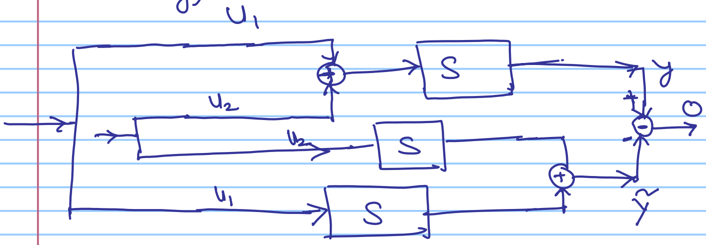
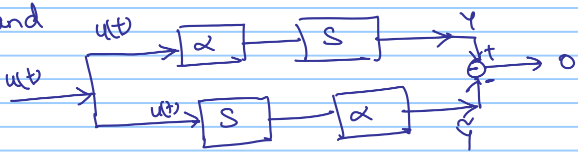
Pictorially,



$y = \tilde{y}$

and



$y = \tilde{y}$

Example:

System: $\left[\dfrac{dV_c}{dt} + \dfrac{1}{RC} V_c = \dfrac{u(t)}{RC} \; ; \; V_c(0) = 0 \right.$

and $\left. y(t) = C\dfrac{dV_c}{dt} \right]$.

- Suppose System is provided with input $u_1(t)$ and the corresponding output is $y_1(t)$ then

$$\frac{dV_{c,1}}{dt} + \frac{1}{RC} V_{c,1} = \frac{u_1(t)}{RC} \; ; \; V_{c,1}(0) = 0$$

$$y_1(t) = C\frac{dV_{c,1}}{dt}$$

- Suppose System is provided with input $u_2(t)$ and the corresponding output is $y_2(t)$ then

$$\frac{dV_{c,2}}{dt} + \frac{1}{RC} V_{c,2} = \frac{u_2(t)}{RC} \; ; \; V_{c,2}(0) = 0$$

$$y_2(t) = C\frac{dV_{c,2}}{dt}$$

→ Suppose the input is $u = (\alpha_1 u_1 + \alpha_2 u_2)$. Then

The output $y(t)$ is given by

$$\left.\begin{aligned}\frac{dV_c}{dt} + \frac{1}{RC} V_c = \alpha_1 u_1 + \alpha_2 u_2 \quad ; V_c(0) = 0 \\ \text{and} \quad y(t) = C\frac{dV_c}{dt}\end{aligned}\right\} \text{--} \; (*)$$

We claim that $V_c(t) = \alpha_1 V_{c,1} + \alpha_2 V_{c,2}$ satisfies $(*)$. Lets check this.

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c =$$

$$= \frac{d(\alpha_1 V_{c,1} + \alpha_2 V_{c,2})}{dt} + \frac{1}{RC}\left[V_{c,1} + V_{c,2}\right]$$

$$= \frac{d}{dt}(\alpha_1 V_{c,1}) + \frac{d}{dt}\alpha_2(V_{c,2}) + \frac{V_{c,1}}{RC} + \frac{V_{c,2}}{RC}$$

$$= \underbrace{\alpha_1 \frac{dV_{c,1}}{dt} + \frac{V_{c,1}}{RC}}_{\alpha_1 u_1} + \underbrace{\alpha_2 \frac{dV_{c,2}}{dt} + \frac{V_{c,2}}{RC}}_{\alpha_2 u_2}$$

$$= \alpha_1 u_1 + \alpha_2 u_2.$$

Also, $V_c(0) = (\alpha_1 V_{c,1} + \alpha_2 V_{c,2})(0)$

$$= \alpha_1 V_{c,1}(0) + \alpha_2 V_{c,2}(0)$$

$$= 0 + 0 = 0$$

$\therefore$ With $V_c = \alpha_1 V_{c,1} + \alpha_2 V_{c,2}$

$V_c(0) = 0$ and

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c(t) = \alpha_1 u_1 + \alpha_2 u_2$$

There the output of the system with input $\alpha_1 u_1 + \alpha_2 u_2$ is

$$y(t) = C\frac{dV_c}{dt} = C\frac{d}{dt}(\alpha_1 V_{c,1} + \alpha_2 V_{c,2})$$

$$= \alpha_1 C\frac{d}{dt} V_{c,1} + \alpha_2 \frac{d}{dt} V_{c,2}$$

$$= \alpha_1 y_1 + \alpha_2 y_2$$

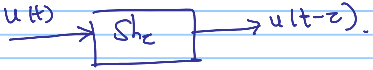
$$\therefore \quad \underbrace{S(\alpha_1 u_1 + \alpha_2 u_2)}_{y(t)} = \alpha_1 \underbrace{(S u_1)}_{y_1} + \alpha_2 \underbrace{(S u_2)}_{y_2}.$$

This proves the linearity of the system S described by ordinary differential equations.

Time-Invariance:

Definition: The Shift operator:

$$(Sh_\tau u)(t) \triangleq u(t-\tau)$$



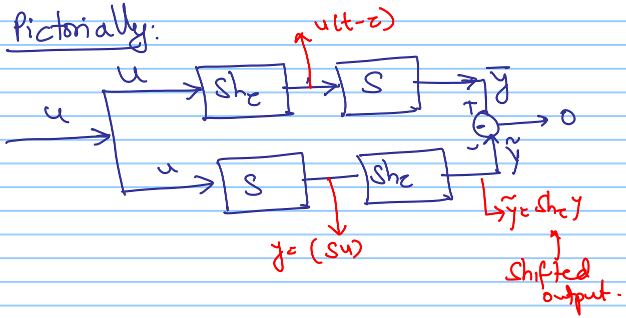
(*) Verify that the Shift operator is Linear

Let S be a System. If

$$S \cdot Sh_\tau = Sh_\tau \cdot S \qquad \text{then}$$

S is said to be time-invariant.

Pictorially:



Example

Back to the RC Circuit

- Suppose the input is $u(t)$ and the corresponding output is $y(t)$. Then let $v_c(t)$ be such that

$$\frac{dv_c}{dt} + \frac{1}{RC} v_c = \frac{u}{RC} \quad ; v_c(0) = 0$$

$$y(t) = C\frac{dv_c}{dt}.$$

- Suppose the input is $(Sh_\tau u)(t) = u(t-\tau)$. τ is a constant.

Consider $\tilde{v}_c(t) = v_c(t-\tau)$.

Note that

$$\frac{d\tilde{v}_c(t)}{dt} = \frac{dv_c(t-\tau)}{dt}$$

$$= \frac{dv_c(\sigma)}{d\sigma} \quad ; \text{ let } \sigma \text{ be } \sigma = t-\tau.$$

$$\therefore \frac{d\tilde{v}_c(t)}{dt} + \frac{1}{RC}\tilde{v}_c(t) = \frac{dv_c(\sigma)}{d\sigma} + \frac{1}{RC}v_c(t-\tau)$$

$$= \dot{v}_c(\sigma) + \frac{1}{RC}v_c(\sigma)$$

$$= u(\sigma)$$

$$= u(t-\tau)$$

$\therefore \frac{d\tilde{v}_c(t)}{dt} + \frac{1}{RC}\tilde{v}_c(t) = u(t-\tau)$;

Also, $\tilde{y}(t) = C\frac{d\tilde{v}_c(t)}{dt}$

$= C\frac{dv_c(t-\tau)}{dt}$

$= C\frac{dv_c(\sigma)}{d\sigma}$

$= y(\sigma)$

$= y(t-\tau)$.

Therefore

$u(t) \xrightarrow{S} y(t)$

$u(t-\tau) \xrightarrow{S} y(t-\tau)$.

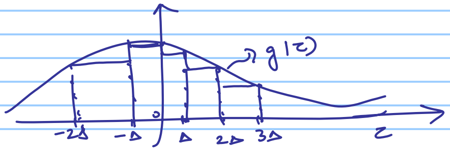
(Shift the input by τ) results in an output that is shifted by τ.

Linearity and time invariance play a very important role in describing a number of systems.

Essentially for linear and time invariant systems determining the output due to one special input suffices to determine the output of any other general input. The special input is the "impulse" input

The Impulse Function:

$$f(t) = \int_{-\infty}^{\infty} g(\tau)\, d\tau$$



$$\int_{-\infty}^{\infty} g(\tau)\, d\tau = g(0)\cdot\Delta + g(-\Delta)\Delta + g(\Delta)\Delta$$
$$+ g(-2\Delta)\Delta + g(2\Delta)\Delta$$
$$+ \cdots$$

$$\cong \sum_{n=-\infty}^{\infty} g(n\Delta)\,\Delta.$$

with $g(\tau) = f(\tau)\,\delta(t-\tau)$

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau) \cong \sum_{n=-\infty}^{\infty} f(n\Delta)\,\delta(t-n\Delta)\,\Delta$$

Thus any function 'u' can be written as a summation of scaled and shifted impulses $\delta(t-n\Delta)$ i.e.

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau = \sum_{n=-\infty}^{\infty} u(n\Delta)\,\delta(t-n\Delta)\,\Delta$$

Impulse Response of a System.

Suppose the output of a linear time invariant system S for an impulse input at time $t=0$ is $h(t)$

$$\delta(t) \longrightarrow \boxed{\;S\;} \longrightarrow h(t)$$

Then consider an arbitrary input $u(t)$

$$u(t) \longrightarrow \boxed{\;S\;} \longrightarrow y(t)$$

Note that

$$u(t) = \sum_{n=-\infty}^{\infty} u(n\Delta)\, \delta(t-n\Delta)\, \Delta.$$

$$\therefore \; (Su)(t) = S\left[\sum_{n=-\infty}^{\infty} u(n\Delta)\, \Delta\, \delta(t-n\Delta)\right]$$

Linearity $\longleftarrow$ $$= \sum_{n=-\infty}^{\infty} \left[u(n\Delta)\, \Delta\, S\,\delta(t-n\Delta)\right]$$

Linearity $\longleftarrow$ $$= \sum_{n=-\infty}^{\infty} \left[u(n\Delta)\, \Delta\, S \cdot Sh_{n\Delta}\,\delta(t)\right]$$

time-invariance $\longleftarrow$ $$= \sum_{n=-\infty}^{\infty} \left[u(n\Delta)\, \Delta\, Sh_{n\Delta}(S\delta)(t)\right]$$

Impulse response is $h(t)$

$$= \sum_{n=-\infty}^{\infty} u(n\Delta)\, \Delta\, S h_{n\Delta}^{h(t)}$$

$$= \sum_{n=-\infty}^{\infty} u(n\Delta)\, \Delta\, h(t-n\Delta).$$

$$\cong \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau.$$

$\therefore$ The output due to an arbitrary input $u(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$$

where $h(t) = (S\delta)(t)$ is the impulse response of the time-invariant system S.