Linear Control Systems: Week 1.

Note Title 9/9/2009

Modeling:

The primary modeling methodologies are

① Physical modelling

② Input-output based black-box modeling

In many practical scenarios a mix of physical and black-box modeling techniques are employed. We will first present examples of physical modeling; followed by black-box modeling techniques; followed by the relationships of the two modeling methodologies.

Physical modeling:

→ Use of physics based laws to obtain mathematical descriptions of the system under study

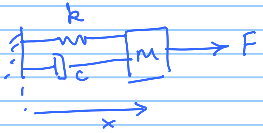
Mechanical Systems

→ Mostly governed by Newtons laws of motion particularly the last two laws.

→ Consider any body with mass 'm' that is acted upon by a net force $\vec{F}$.
Then
$$\vec{F} = m\vec{a}$$
where $\vec{a}$ is the acceleration of the body m (2nd law)

→ Every action has an equal and opposite reaction. (Third law)

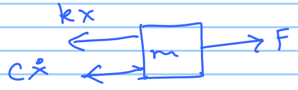
Example 1:



→ A linear (Hookean) spring exerts a force kx in the direction opposite to the direction of the displacement x.

→ A viscous damper exerts a force $c\dot{x}$ in the direction opposite to the displacement x.

Free-body diagram:



Net force on the mass m in the direction of motion is

$$F_{net} = F - kx - c\dot{x}$$

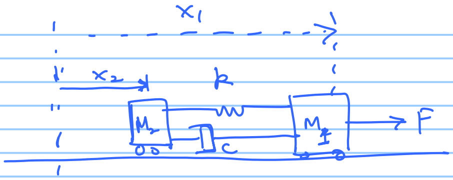
∴ from Newtons laws

$$m\frac{d^2x}{dt^2} = F - kx - c\dot{x}$$

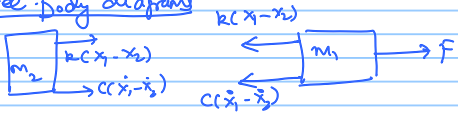
$$\Rightarrow \boxed{m\frac{d^2x}{dt^2} + c\dot{x} + kx = F.}$$

The above differential equation is the mathematical description of a spring-mass-damper system.

Example 2



Free body diagrams



Net force acting on mass m_1 is

$$F - k(x_1-x_2) - c(\dot{x}_1-\dot{x}_2)$$

$$\therefore \quad m_1 \frac{d^2x_1}{dt^2} = F - k(x_1-x_2) - c(\dot{x}_1-\dot{x}_2)$$

$$\Rightarrow \quad m_1 \frac{d^2x_1}{dt^2} + c\dot{x}_1 - c\dot{x}_2 + kx_1 - kx_2 = F$$

Net force acting on mass m_2 is

$$k(x_1-x_2) + c(\dot{x}_1-\dot{x}_2)$$

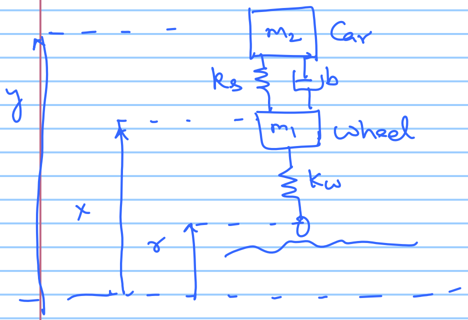
$$\therefore \quad m_2 \frac{d^2x_2}{dt^2} = k(x_1-x_2) + c(\dot{x}_1-\dot{x}_2)$$

$$\Rightarrow \quad m_2 \frac{d^2x_2}{dt^2} - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 = 0$$

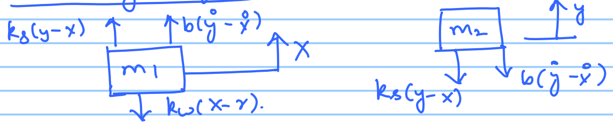
∴ The mathematical model describing the two mass system is

$$\boxed{\begin{aligned} m_1 \frac{d^2x_1}{dt^2} + c\dot{x}_1 - c\dot{x}_2 + kx_1 - kx_2 &= F \\ m_2 \frac{d^2x_2}{dt^2} - c\dot{x}_1 + c\dot{x}_2 - kx_1 + kx_2 &= 0 \end{aligned}}$$

Example 3: A quarter Car model:



Free body diagrams:



Net force on m_1 in the direction of x is

$$k_s(y-x) + b(\dot{y}-\dot{x}) - k_w(x-r).$$

$$\therefore \quad m_1 \frac{d^2x}{dt^2} = k_s(y-x) + b(\dot{y}-\dot{x}) - k_w(x-r).$$

$$\Rightarrow \quad m_1 \frac{d^2x}{dt^2} + b\dot{x} - b\dot{y} + k_s x - k_s y + k_w x = k_w r.$$

Net force on the 'car' in the direction y is

$$-k_s(y-x) - b(\dot{y}-\dot{x})$$

$$\therefore \quad m_2 \frac{d^2y}{dt^2} = -k_s(y-x) - b(\dot{y}-\dot{x})$$

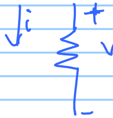
Thus, the mathematical description of the quarter car model is

$$\boxed{\begin{aligned} & m_1 \frac{d^2x}{dt^2} + b\dot{x} - b\dot{y} + k_s x - k_s y + k_w x = k_w r \\ & m_2 \frac{d^2y}{dt^2} + k_s(y-x) + b(\dot{y}-\dot{x}) = 0 \end{aligned}}$$

The above are examples of translational motion. We will provide examples of rotational motion later.

Electrical Systems

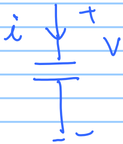
resistor:



$v = iR$

"The voltage drop along the direction of the current is iR. where R is the resistance"

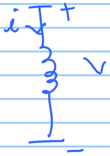
Capacitor



$$i = C\frac{dv}{dt}$$

where v is the voltage drop in the direction of the current

inductor



$$v = L\frac{di}{dt}.$$

where v is the voltage drop in the direction of the current.

Two Laws:

Kirchoff's Current Law (KCL)

The algebraic sum of currents at a node is zero.

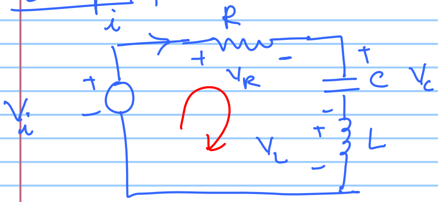
we will adopt a convention that with respect to a node, a current leaving the node is positive and a current entering the node is negative

Kirchoff's Voltage law (KVL):

The algebraic sum of all voltages taken around a closed path is zero.

→ we will adopt the convention that a voltage drop along the path being traversed is positive and a voltage rise is negative.

Example 4:



Applying KVL we obtain

$$-V_i + V_R + V_c + V_L = 0$$

with

$$V_R = iR$$

$$C\frac{dV_c}{dt} = i$$

$$L\frac{di}{dt} = V_L$$

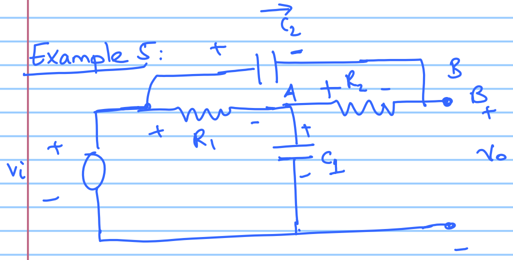
$$\Rightarrow \quad V_R = RC\frac{dV_c}{dt}$$

$$V_L = L\frac{di}{dt} = LC\frac{d^2V_c}{dt^2}$$

$$\therefore \quad -V_i + RC\frac{dV_c}{dt} + V_c + LC\frac{d^2V_c}{dt^2} = 0$$

$$\Rightarrow \quad \boxed{LC\frac{d^2v_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = V_i}$$

<u>Example 5:</u>



<u>Node A:</u>

$$-\left(\frac{V_i - V_A}{R_1}\right) - \frac{(V_B - V_A)}{R_2} + C_1\frac{dV_A}{dt} = 0$$

<u>Node B:</u>

$$+\frac{V_B - V_A}{R_2} - C_2\frac{d(V_i - V_B)}{dt} = 0$$

Let $V_{C_1} = V_A$; $V_{C_2} = V_i - V_B$.

$V_B = V_i - V_{C_2}$

$$\Rightarrow \quad C_1 \frac{dV_{C_1}}{dt} = \frac{V_i - V_A}{R_1} + \frac{V_B - V_A}{R_2}$$

$$= \frac{V_i - V_{C_1}}{R_1} + \frac{V_i - V_{C_2} - V_{C_1}}{R_2}$$

$$= \frac{V_i}{R_1} + \frac{V_i}{R_2} - \frac{V_{C_1}}{R_1} - \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2}$$

$$= V_i\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - V_{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{V_{C_2}}{R_2}$$

$$\therefore \frac{dV_{C_1}}{dt} = \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_{C_1} - \frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_{C_1} - \frac{V_{C_2}}{R_2}$$

Illy

$$\frac{dV_{C_2}}{dt} = \frac{-1}{C_2R_2}V_{C_1} - \frac{1}{C_2R_2}V_{C_2} + \frac{1}{C_2R_2}V_i$$

Rotational dynamics:

I = moment of Inertia

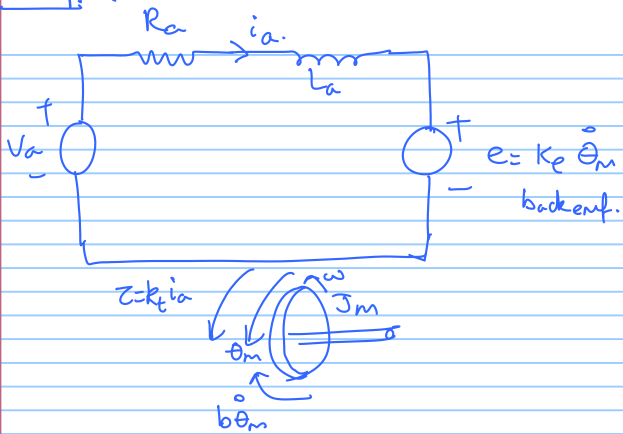
θ = angular position

The main law is

$$I\frac{d^2\theta}{dt^2} = \tau$$

where τ is the net torque acting

Example 6 :



$$-V_a + iR_a + L_a \frac{di}{dt} + K_e \dot{\theta}_m = 0$$

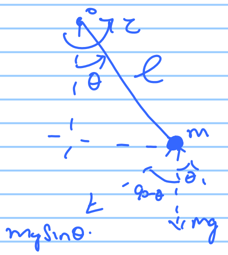
and

$$J_m \ddot{\theta}_m = k_t i_a - b\dot{\theta}_m + \omega$$

$$\therefore \boxed{J_m \ddot{\theta}_m + b\dot{\theta}_m = k_t i_a + \omega} \quad - - \quad (1)$$

$$K_e \dot{\theta}_m + L_a \frac{di_a}{dt} + R_a i_a = V_a \quad - - \quad (2)$$

Example 7



applied torque $= \tau$

restoring moment $= -(mg\sin\theta\,\ell)$

Net torque $= \tau - mg\ell\sin\theta$

$\therefore \quad I\ddot{\theta} = \tau - mg\ell\sin\theta$

$I\ddot{\theta} + mg\ell\sin\theta = \tau$

$\Rightarrow \quad m\ell^2\ddot{\theta} = \tau - mg\ell\sin\theta$

$\Rightarrow \quad \boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = \tau}$

Thus, we have seen that many systems admit mathematical descriptions that are ordinary differential equations