Frequency Response of LTI systems

Friday, October 23, 2009
9:43 AM

Frequency Response of LTI Systems:

- We have seen that when a linear-time-invariant causal system that is stable is subjected to a sinusoidal input the steady state output is also a sinusoid of the same frequency as the input

- Suppose the transfer function of such a system is $H(s)$ the for a sinusoidal input

$$u(t) = \sin \omega t$$

the steady state output is

$$y(t) = |H(j\omega)| \sin(\omega t + \angle H(j\omega))$$

Sinusoidal Response

Friday, October 23, 2009
9:52 AM

(*) Suppose the input is

$\tilde{u}(t) = A \sin(\omega t + \theta)$

$= A \sin\theta \cos\omega t + A \cos\theta \sin\omega t$

- The output of the system with input $\cos\omega t = \frac{1}{\omega} \frac{d}{dt} \sin\omega t$

will be

$\frac{1}{\omega} \frac{d}{dt} |H(j\omega)| \sin[\omega t + \angle H(j\omega)]$

follows from linearity

$= \frac{1}{\omega} |H(j\omega)| \omega \cos[\omega t + \angle H(j\omega)]$

$= |H(j\omega)| \cos[\omega t + \angle H(j\omega)]$

- The output of the system when the input is

$\tilde{u} = A \sin\theta \cos\omega t + A \cos\theta \sin\omega t$ is

$\tilde{y} = A \sin\theta |H(j\omega)| \cos(\omega t + \angle H(j\omega)) +$
$A \cos\theta |H(j\omega)| \sin[\omega t + \angle H(j\omega)]$

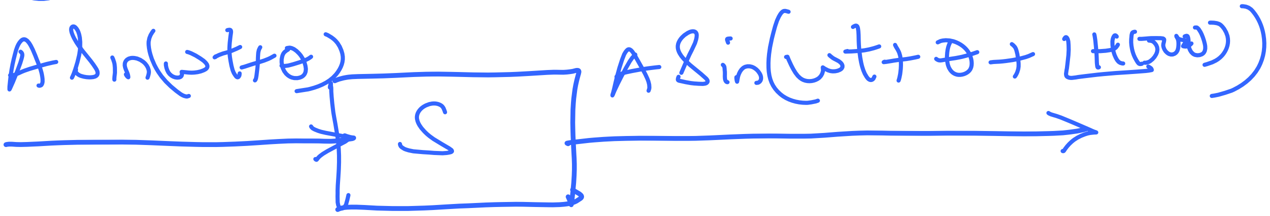
$= |H(j\omega)| [A \sin\theta \cos(\omega t + \angle H(j\omega)) +$
$A \cos\theta \sin(\omega t + \angle H(j\omega))$

Friday, October 23, 2009
9:57 AM

$$\therefore \tilde{y} = A|H(j\omega)| \left[\sin\theta \cos(\omega t + \angle H(j\omega)) + \cos\theta \sin(\omega t + \angle H(j\omega)) \right]$$

$$= A|H(j\omega)| \sin(\omega t + \theta + \angle H(j\omega)).$$

(*) Thus



Conclusion:

(*) For any sinusoidal input with frequency ω, the output is a sinusoid of the same frequency ω and phase $\angle H(j\omega)$ and an amplitude scaled by $|H(j\omega)|$

Summary

Friday, October 23, 2009
10:00 AM

Frequency Response of a linear time invariant stable Causal System:

* Let $H(s)$ be the transfer function of the system
  - Laplace transform of the impulse response is $H(s)$

* The frequency response of the system is characterized by the complex number

$$H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)}$$

* In most situations the magnitude gain $|H(j\omega)|$ and phase $\angle H(j\omega)$ are characterized by Bode plots.

Building Blocks

Friday, October 23, 2009
10:01 AM

Frequency Response of a linear time invariant stable Causal System:

- Let $H(s)$ be the transfer function of the system
  - laplace transform of the impulse response is $H(s)$
- The frequency response of the system is characterized by the complex number

$$H(j\omega) = |H(j\omega)| \, e^{j\angle H(j\omega)}$$

- In most situations the magnitude gain $|H(j\omega)|$ and phase $\angle H(j\omega)$ are characterized by Bode plots.

BODE PLOTS

Friday, October 23, 2009
10:02 AM

Bode Plots:

Bode plot a system with transfer function $H(s)$ is given by

two plots

(1) Plot of $20\lg_{10}|H(j\omega)|$ vs $\lg_{10}\omega$

(2) Plot of $\underline{/H(j\omega)}$ vs $\lg_{10}\omega$

* The gain plot $20\lg_{10}|H(j\omega)|$ is termed decibel (db).

- $20\lg_{10}1 = 0$ db
- $20\lg_{10}10 = 20$ db.

Bode plot of a constant gain

Friday, October 23, 2009
10:04 AM

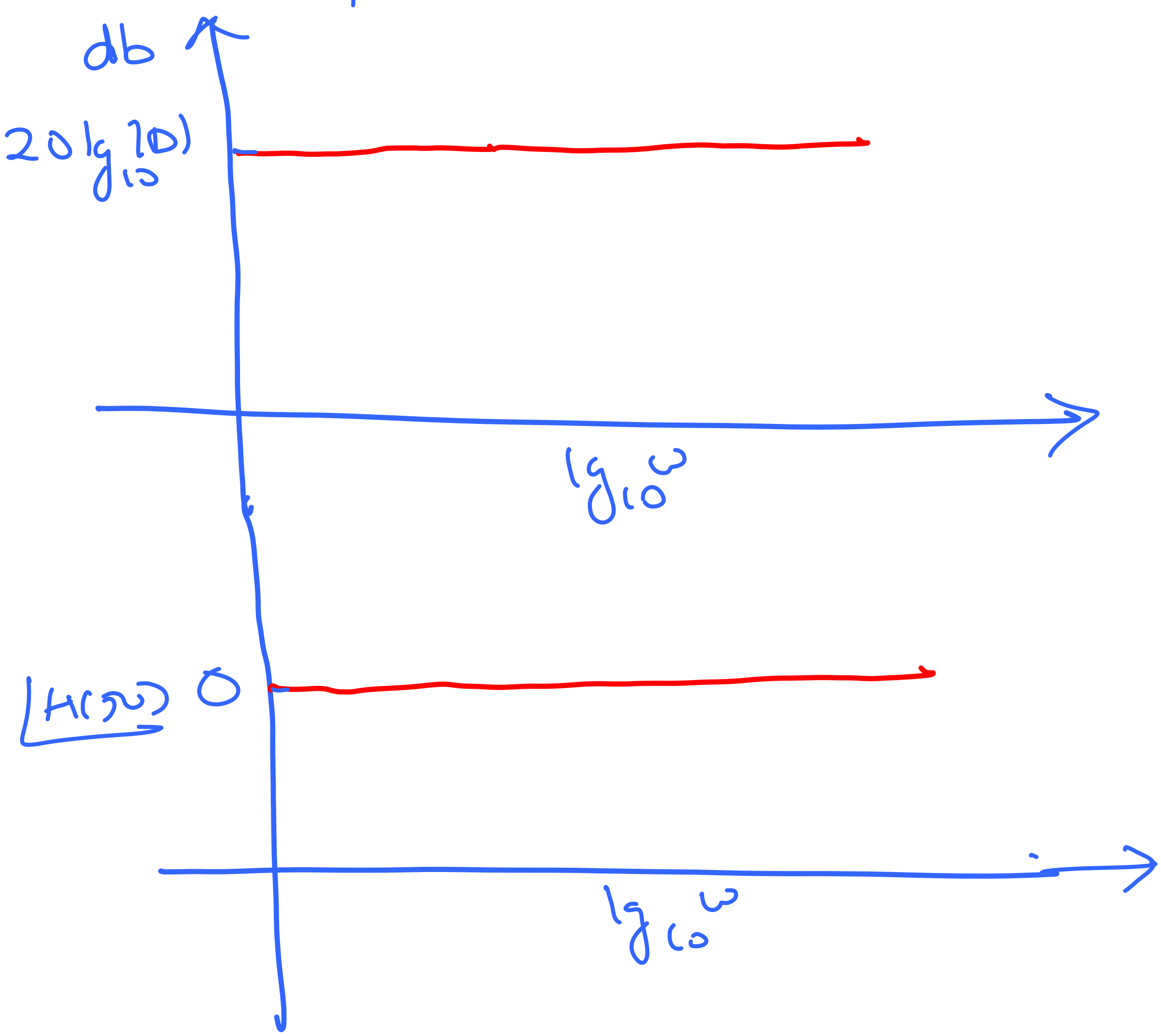
Frequency response of a constant gain

- $H(s) = D > 0.$

Then

- $20\lg|H(j\omega)| = 20\lg_{10}|D|$
- $\angle H(j\omega) = 0$

The bode plot



Real first order pole

Friday, October 23, 2009
1:20 PM

Bode plot of a real first order pole:

$$H(s) = \frac{1}{1+\frac{s}{p}} \quad ; \ p > 0$$

$$\Rightarrow \quad H(j\omega) = \frac{1}{1+\frac{j\omega}{p}}$$

$$20\log_{10}|H(j\omega)| = 20\log_{10}\left|\frac{1}{\left(1+\frac{j\omega}{p}\right)}\right|^2$$

$$= 20\log_{10}\frac{1}{\left(1+\frac{\omega^2}{p^2}\right)^{1/2}}$$

$$= -20\log_{10}\left(1+\frac{\omega^2}{p^2}\right)^{1/2}$$

$$\angle H(j\omega) = -\angle\left(1+\frac{j\omega}{p}\right)$$

Asymptotes

Saturday, October 24, 2009
12:21 PM

Asymptotes: $H(j\omega) = \frac{1}{1+\frac{\omega^2}{p^2}} - j\frac{\omega/p}{1+\frac{\omega^2}{p^2}}$.

Case 1: $\omega << p$. This is the low frequency scenario.

$$20\lg_{10}|H(j\omega)| = -20\lg\left(1+\frac{\omega^2}{p^2}\right)^{1/2} \quad \omega << p$$

$$\approx -20\lg 1 = 0$$

Case 2: $\omega >> p$. This is the high frequency scenario.

$$20\lg_{10}|H(j\omega)| = -20\lg\left(1+\frac{\omega^2}{p^2}\right)^{1/2}; \omega >> p$$

$$\cong -20\lg\left(\frac{\omega^2}{p^2}\right)^{1/2}$$

$$= -20\lg\frac{\omega}{|p|}$$

Asymptotes

Saturday, October 24, 2009
12:22 PM

Case 3 : $\omega = \omega_0$; the break frequency

$$20\log_{10}|H(j\omega)| = -20\log_{10}\sqrt{1+\frac{\omega^2}{p^2}}$$

$$\simeq -20\log_{10}\sqrt{2}$$

$$= -3.01 \text{ dB}.$$

Asymptotes(phase)

Saturday, October 24, 2009
12:23 PM

Phase:

$$H(j\omega) = \frac{1}{1+\frac{\omega^2}{p^2}} - j\frac{\omega/p}{1+\omega^2/p^2}$$

Case 1: $\omega << p$

$$H(j\omega) \approx 1 \Rightarrow \angle H(j\omega) = 0$$

Case 2: $\omega >> p$

$$H(j\omega) \approx -j\frac{\omega/p}{\omega^2/p^2}$$

$$= -j\, p/\omega$$

$$\angle H(j\omega) = -90^\circ$$

Case 3: $\omega = p$

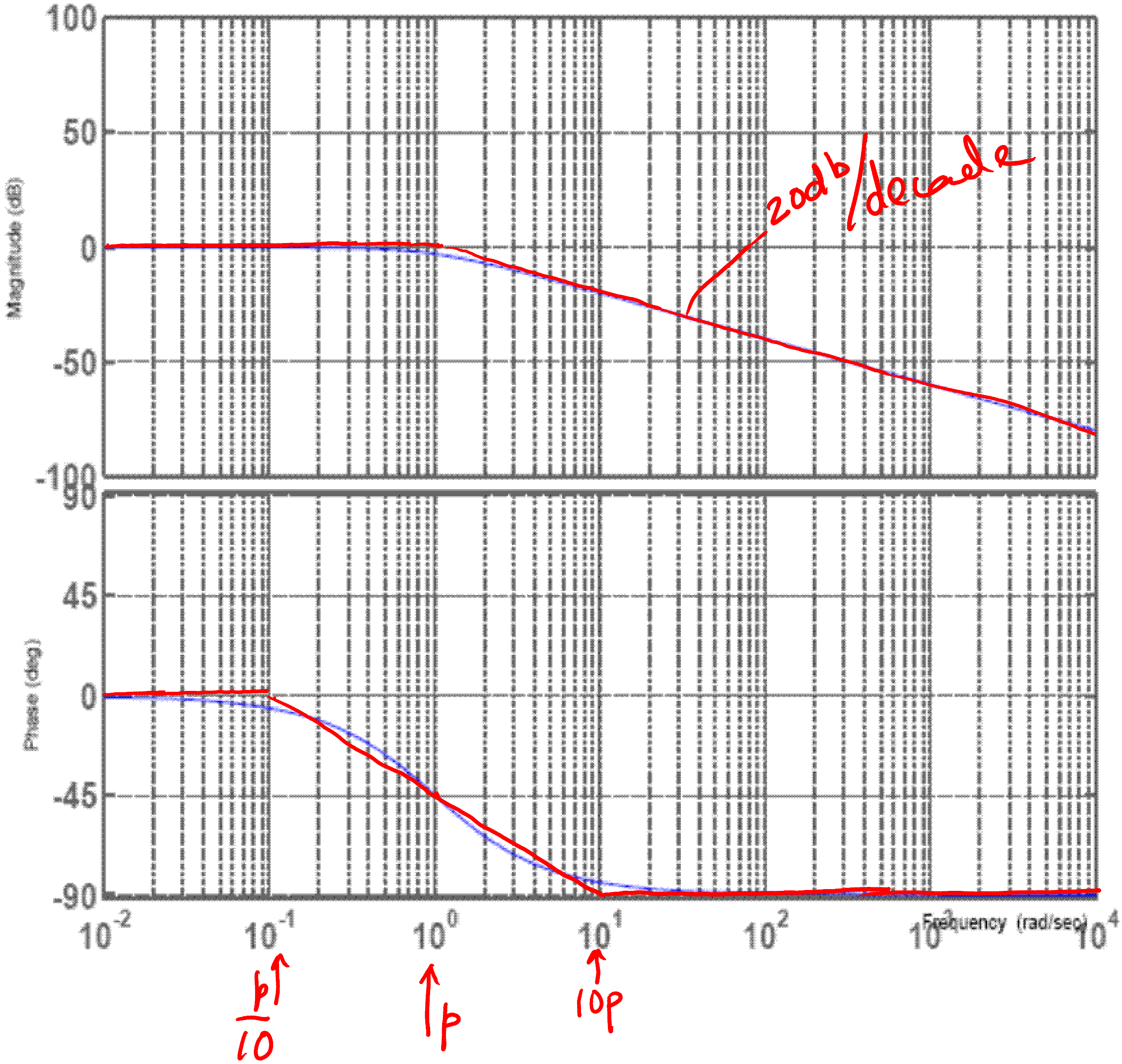
$$H(j\omega) = \frac{1}{2} - j/2$$

$$\Rightarrow \angle H(j\omega) = -45^\circ$$

Asymptotes(plot)

Saturday, October 24, 2009
12:25 PM

Asymptotes for $\frac{1}{s+1}$.



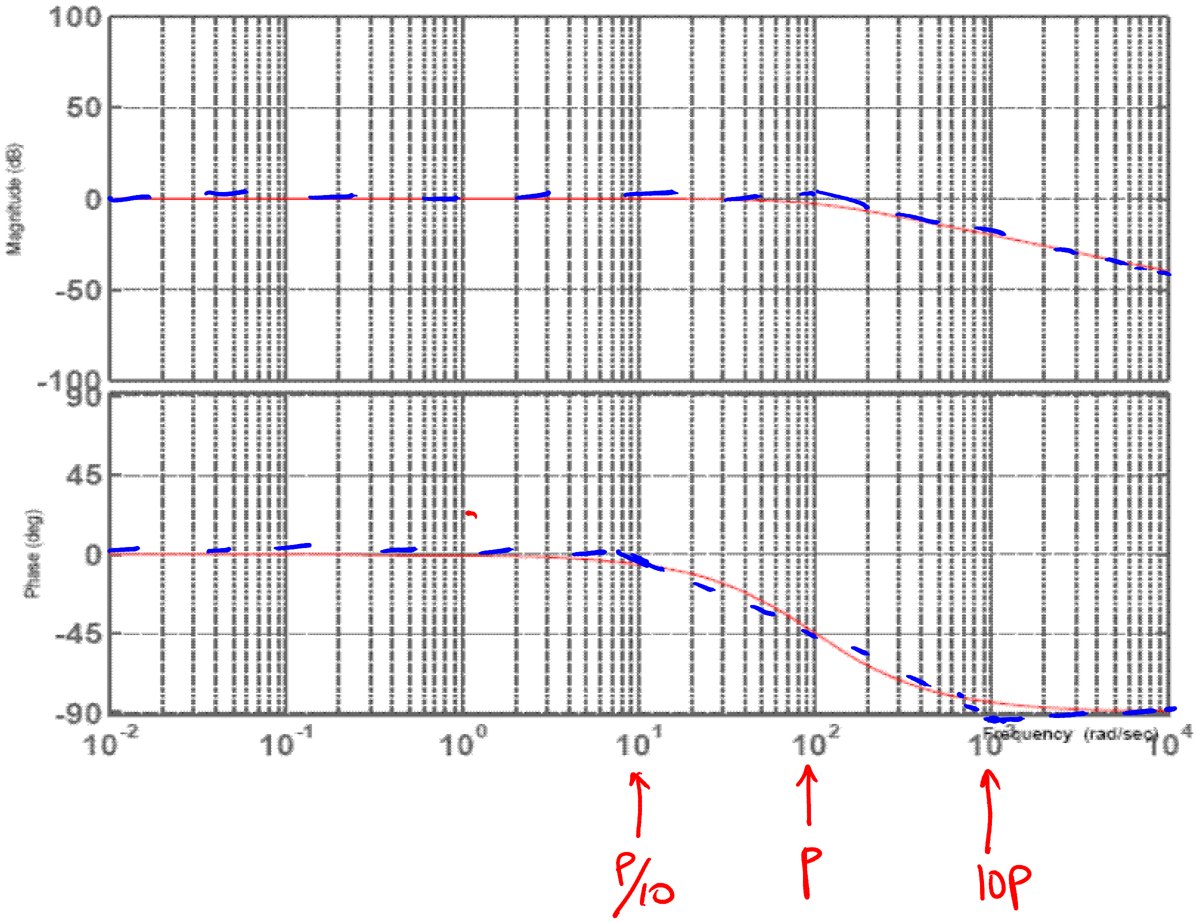
Pole at 100 rad/sec

Saturday, October 24, 2009
12:41 PM

Pole at $s = 100$

Bode plot of $\frac{1}{1+s/100}$

(a) Break frequency of $p_2 = 100$.



Bode plot of a real zero

Saturday, October 24, 2009
12:32 PM

$$H(s) = 1 + s/z \quad ;$$

$$H(j\omega) = 1 + \frac{j\omega}{z} \quad ; \quad z > 0$$

$$\Rightarrow \quad |H(j\omega)| = \sqrt{1 + \frac{\omega^2}{z^2}}$$

$$\Rightarrow \quad 20\log_{10}|H(j\omega)| = 20\log_{10}\sqrt{1 + \frac{\omega^2}{z^2}}$$

$$\angle H(j\omega) = \tan^{-1}\frac{\omega}{z} .$$

<u>Case 1</u>: $\omega \ll z$

$$H(j\omega) \approx 1$$

$$20\log_{10}|H(j\omega)| = 0$$

$$\angle H(j\omega) = 0$$

Bode plot of a real zero

Saturday, October 24, 2009
12:36 PM

Case 2: $\omega \gg z$

$$H(j\omega) \approx j\frac{\omega}{z}$$

$$\Rightarrow 20\log_{10}|H(j\omega)| = 20\log_{10}\frac{\omega}{z}$$

$$\angle H(j\omega) = 90^\circ.$$

Case 3: $\omega = z$

$$H(j\omega) = 1 + j\cdot 1$$

$$\Rightarrow 20\log_{10}|H(j\omega)| = 20\log\sqrt{2} \simeq 3\,db$$

$$\angle H(j\omega) = 45^\circ.$$

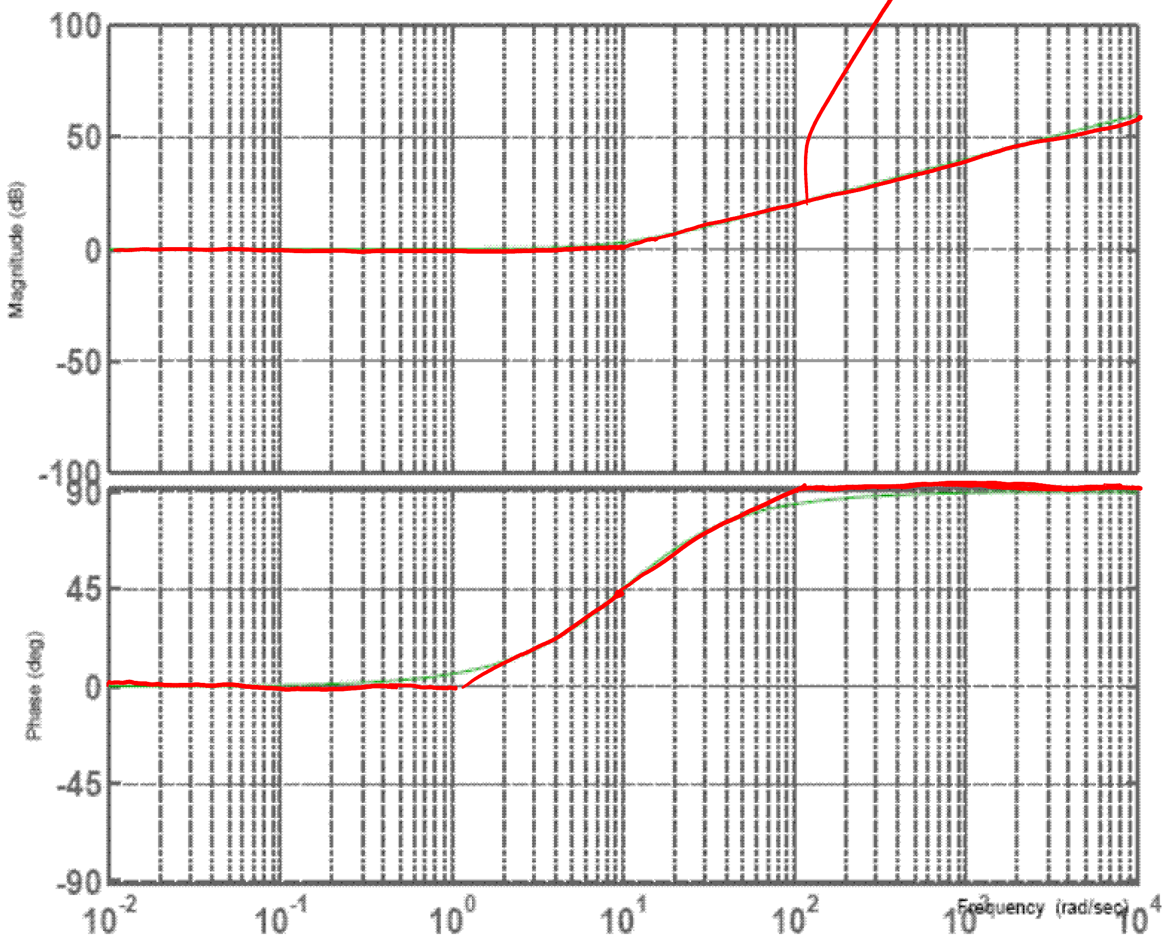
Zero at 10 Hz

Saturday, October 24, 2009
12:38 PM

Example:

$$H(s) = 1 + \frac{s}{10} \quad ; \; z = 10$$

20db/decade



z/10   z   10z

Combination of simple zeros and poles

Saturday, October 24, 2009
12:47 PM

Consider a transfer function that has no complex poles and no complex zeros given by

$$G(s) = A \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where

$-z_1, -z_2 \cdots -z_m$ are the zeros and

$-p_1, -p_2, \cdots -p_n$ are the poles of the transfer function $G(s)$.

- We will also assume that

$|z_1| \leq |z_2| \leq |z_3| \cdots \leq |z_m|$

$|p_1| \leq |p_2| \leq |p_3| \cdots \leq |p_n|$.

Standard form

Saturday, October 24, 2009

1:01 PM

Step1: Convert the transfer function to the standard form.

$$G(s) = A\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= A\frac{z_1\left(1+\frac{s}{z_1}\right)z_2\left(1+\frac{s}{z_2}\right)\cdots z_m\left(1+\frac{s}{z_m}\right)}{p_1\left(1+\frac{s}{p_1}\right)p_2\left(1+\frac{s}{p_2}\right)\cdots p_n\left(1+\frac{s}{p_n}\right)}$$

$$= \underbrace{\frac{A z_1 z_2\cdots z_m}{p_1 p_2\cdots p_n}}_{B}\frac{\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\cdots\left(1+\frac{s}{z_m}\right)}{\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\cdots\left(1+\frac{s}{p_n}\right)}$$

$$= B\frac{\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\cdots\left(1+\frac{s}{z_m}\right)}{\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\cdots\left(1+\frac{s}{p_n}\right)}$$

where $B, z_1, z_2\cdots z_m, p_1, p_2\cdots, p_n$ are all real numbers.

Magnitude

Saturday, October 24, 2009
1:05 PM

Step 2: Magnitude plot:

Note that

$$|G(j\omega)| = |B| \frac{\left|1+\frac{j\omega}{z_1}\right| \left|1+\frac{j\omega}{z_2}\right| \cdots \left|1+\frac{j\omega}{z_m}\right|}{\left|1+\frac{j\omega}{p_1}\right| \left|1+\frac{j\omega}{p_2}\right| \cdots \left|1+\frac{j\omega}{p_n}\right|}$$

$\therefore$ $20\lg_{10}|G(j\omega)|$

$$= 20\lg_{10}|B| + 20\lg_{10}\left|1+\frac{j\omega}{z_1}\right| + 20\lg_{10}\left|1+\frac{j\omega}{z_2}\right| + \cdots + 20\lg_{10}\left|1+\frac{j\omega}{z_m}\right| + 20\lg_{10}\left|\frac{1}{1+\frac{j\omega}{p_1}}\right| + 20\lg_{10}\left|\frac{1}{1+\frac{j\omega}{p_2}}\right| + \cdots + 20\lg_{10}\left|1+\frac{j\omega}{p_n}\right|.$$

Thus, the magnitude Bode plot of G is equal to the sum of the magnitude Bode plots of constant $|B|$ and real zeros of the form $\left|1+\frac{j\omega}{z_i}\right|$ and real poles $\left|\frac{1}{1+\frac{j\omega}{p_i}}\right|$.

Phase

Saturday, October 24, 2009
1:11 PM

Step3: Let $\angle G(j\omega)$ denote the phase of $G(j\omega)$. Note that

$$G(j\omega) = B \frac{\left(1+\frac{j\omega}{z_1}\right)\left(1+\frac{j\omega}{z_2}\right)\cdots\left(1+\frac{j\omega}{z_m}\right)}{\left(1+\frac{j\omega}{p_1}\right)\left(1+\frac{j\omega}{p_2}\right)\cdots\left(1+\frac{j\omega}{p_n}\right)}.$$

In the phasor notation a complex number $x = \alpha + j\beta$ where $\mathrm{Real}(x) = \alpha$ and $\mathrm{Img}(x) = \beta$. is written as

$$x = |x|\, e^{j\angle x}$$

where $|x| = \sqrt{\alpha^2+\beta^2}$ and $\angle x = \tan^{-1}(\beta/\alpha)$

Thus

$$|G(j\omega)|\, e^{j\angle G(j\omega)} = |B|\, e^{j\angle B}\, \left|1+\frac{j\omega}{z_1}\right| e^{j\angle 1+j\omega/z_1} \cdot \left|1+\frac{j\omega}{z_2}\right| e^{j\angle 1+j\omega/z_2} \cdots \left|1+\frac{j\omega}{z_m}\right| e^{j\angle 1+j\omega/z_m}$$

$$\left|\frac{1}{1+\frac{j\omega}{p_1}}\right| e^{j\angle \frac{1}{1+j\omega/p_1}} \cdots \left|\frac{1}{1+\frac{j\omega}{p_n}}\right| e^{j\angle \frac{1}{1+j\omega/p_n}}$$

Phasor form

Saturday, October 24, 2009
1:23 PM

$\therefore |G(j\omega)| e^{j \angle G(j\omega)}$

$$= \frac{|B| \left|1+\frac{j\omega}{z_1}\right| \left|1+\frac{j\omega}{z_2}\right| \cdots \left|1+\frac{j\omega}{z_m}\right|}{\left|1+\frac{j\omega}{p_1}\right| \left|1+\frac{j\omega}{p_2}\right| \cdots \left|1+\frac{j\omega}{p_n}\right|} e^{j\theta}$$

$$\theta = \angle B + \angle 1+\frac{j\omega}{z_1} + \angle 1+\frac{j\omega}{z_2} \cdots + \angle 1+\frac{j\omega}{z_m}$$
$$+ \angle \frac{1}{1+j\omega/p_1} + \cdots + \angle \frac{1}{1+j\omega/p_n}$$

Thus,

$$|G(j\omega)| = \frac{|B| \left|1+\frac{j\omega}{z_1}\right| \left|1+\frac{j\omega}{z_2}\right| \cdots \left|1+\frac{j\omega}{z_m}\right|}{\left|1+\frac{j\omega}{p_1}\right| \left|1+\frac{j\omega}{p_2}\right| \cdots \left|1+\frac{j\omega}{p_n}\right|}$$

$$\angle G(j\omega) = \angle B + \angle 1+\frac{j\omega}{z_1} + \angle 1+\frac{j\omega}{z_2} + \cdots \angle 1+\frac{j\omega}{z_m}$$
$$+ \angle \frac{1}{1+j\omega/p_1} + \angle \frac{1}{1+j\omega/p_2} + \cdot \angle \frac{1}{1+j\omega/p_n}$$

Thus the phase of $G(j\omega)$ is simply the sum of the phases of individual terms.

Summary

Friday, October 23, 2009
1:18 PM

The magnitude part of the Bode plot is obtained by adding the magnitude plots of $1+\frac{s}{z_i}$; $i = 1 \cdots m$

and $\frac{1}{1+\frac{s}{p_i}}$; $i = 1 \cdots n$

and the phase plot similarly can be obtained by adding the phase of $1+\frac{s}{z_i}$ $i = 1 \cdots m$

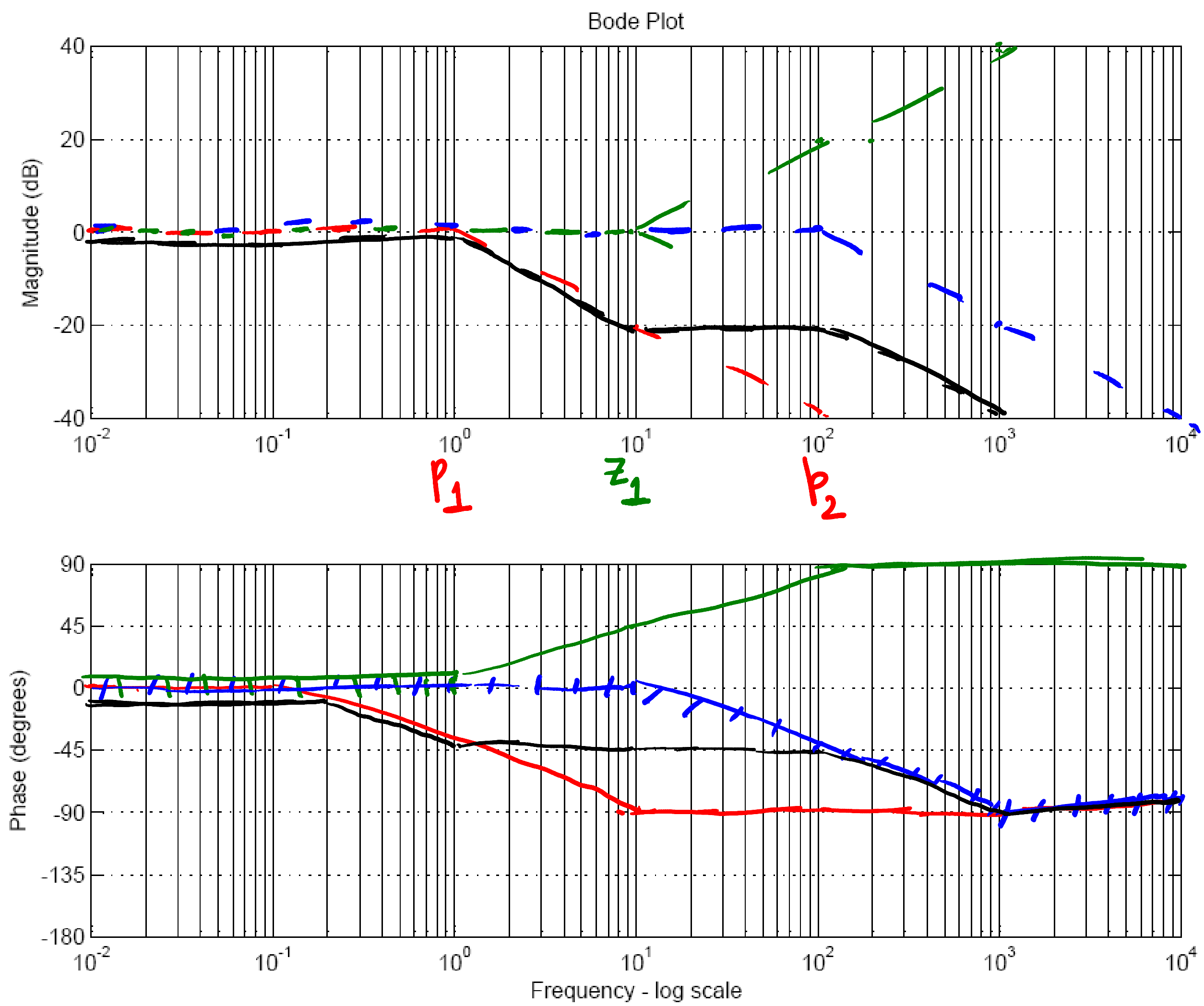
and $\frac{1}{\frac{s}{p_i}+1}$; $i = 1 \cdots n$.

Example

Friday, October 23, 2009

$$H(s) = \frac{\underbrace{1 + s/10}_{z_1}}{\underbrace{(1+s)}_{p_1}\,\underbrace{(1+s/100)}_{p_2}}$$

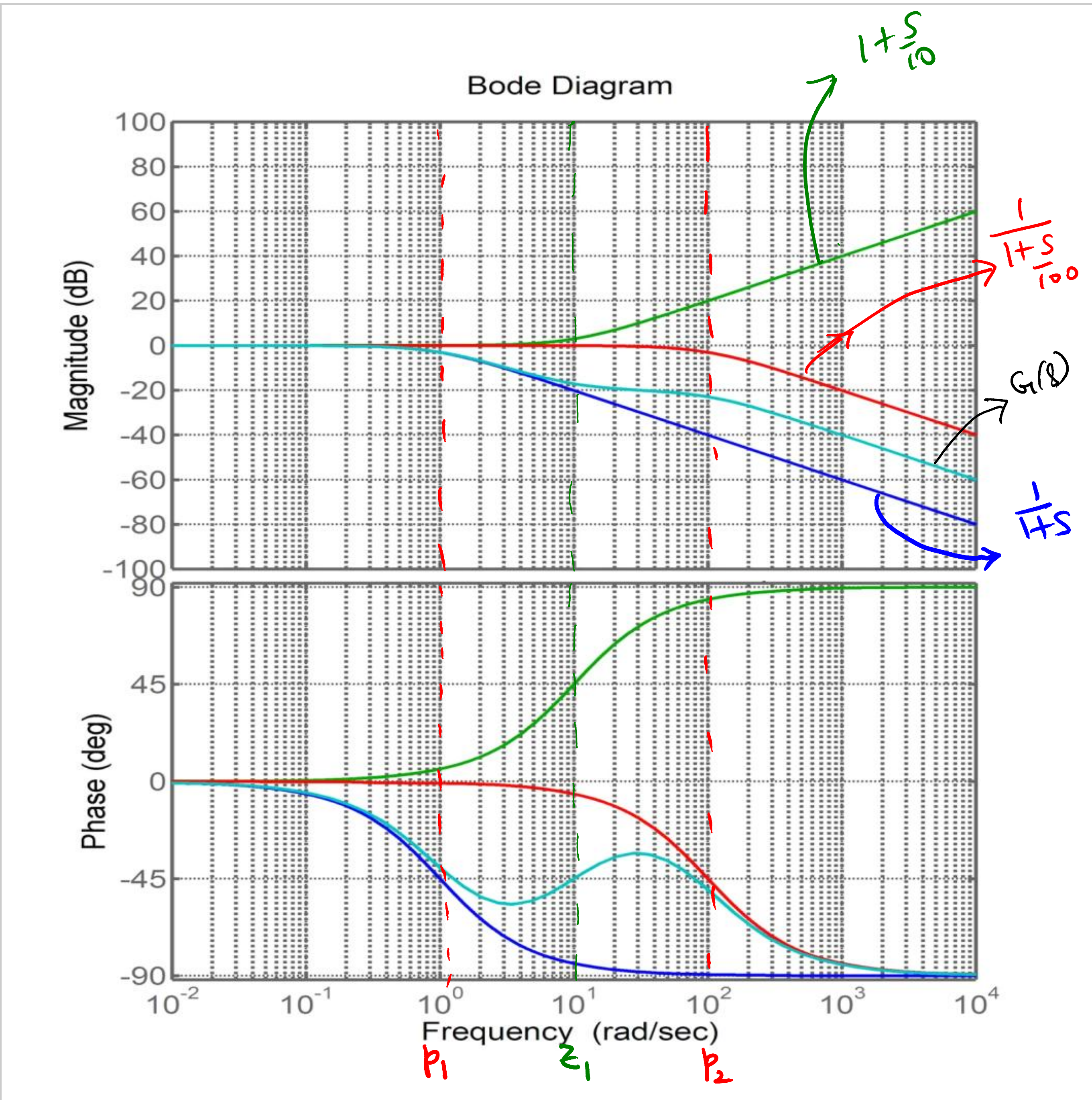
There is a zero at 10 rad/sec
a pole at 1 rad/sec
a pole at 100 rad/sec.



Bode Plot.

Saturday, October 24, 2009
2:39 PM

$$G(s) = \frac{1 + s/10}{(1+s)(1+\frac{s}{100})}$$



Repeated Real zeros

Saturday, October 24, 2009
5:44 PM

Consider $G(s) = \left(1 + \frac{s}{3}\right)^2 = \left(1 + \frac{s}{3}\right)\left(1 + \frac{s}{3}\right)$

that has a repeated zero at

$s = -3$.

The breakpoint is at 3 rad/sec.

Magnitude:

$$G(j\omega) = \left(1 + \frac{j\omega}{3}\right)^2$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\left|1 + \frac{j\omega}{3}\right|^2$$

$$= 40\log_{10}\left|\left(1 + \frac{j\omega}{3}\right)\right|$$

Case 1: $\omega < 3$

$$20\log_{10}|G(j\omega)| \simeq 40\log_{10} 1 = 0$$

Case 2: $\omega > 3$

$$20\log_{10}|G(j\omega)| = 40\log_{10}\left|1 + \frac{j\omega}{3}\right| \simeq 40\log_{10}\left|\frac{\omega}{3}\right|$$

$$= 40\log_{10}|\omega| - 40\log_{10}|3|$$

Repeated zero.

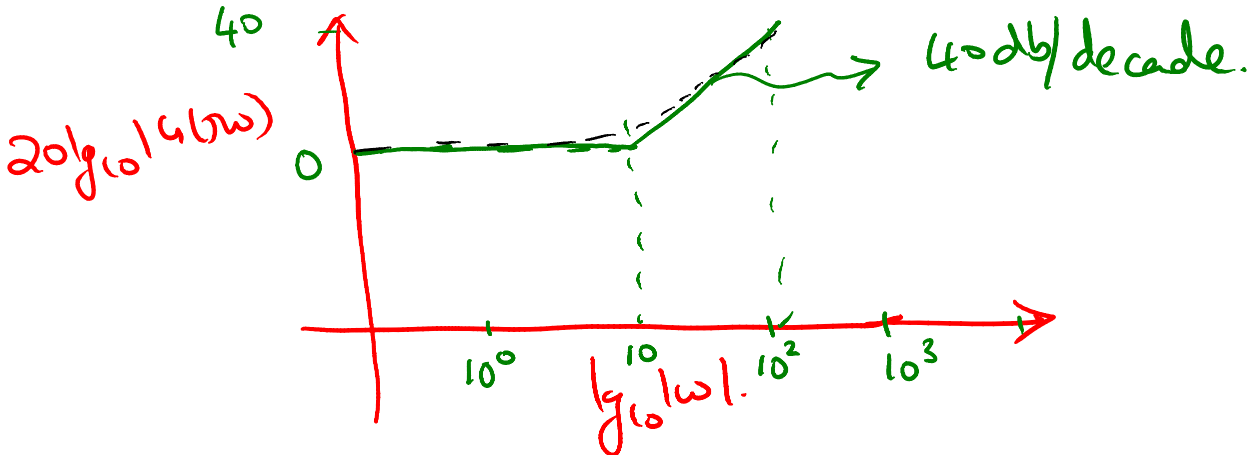
Saturday, October 24, 2009
6:00 PM

Thus, if $\omega > |z|$ then

$$\underbrace{20\log_{10}|G(j\omega)|}_{y} = 40\log_{10}\underbrace{|\omega|}_{x} - 40\log|z|.$$

$$= m\, x + c.$$

Thus, the magnitude part of the bode plot (y) against $\log_{10}|\omega|$ has a slope of 40. (db/decade).



Bode PLOT when $z = 10$.

Repeated zero phase

Saturday, October 24, 2009
6:17 PM

Let $G(s) = \left(\frac{s}{z} + 1\right)^2$

Then $G(j\omega) = \left(1 + \frac{j\omega}{z}\right)^2$

Case 1 : $\omega < z/10$.

$G(j\omega) \approx 1$

$\Rightarrow \quad \angle G(j\omega) = 0.$

Case 2 : $\omega \approx z$

$$G(j\omega) = (1+j)^2$$

$$= 1 + j^2 + 2j$$

$$= 1 - 1 + 2j$$

$$= 2j$$

$\Rightarrow \quad \angle G(j\omega) = \tan^{-1}[\infty] = \frac{\pi}{2}$

Case 3 : $\omega > 10z$

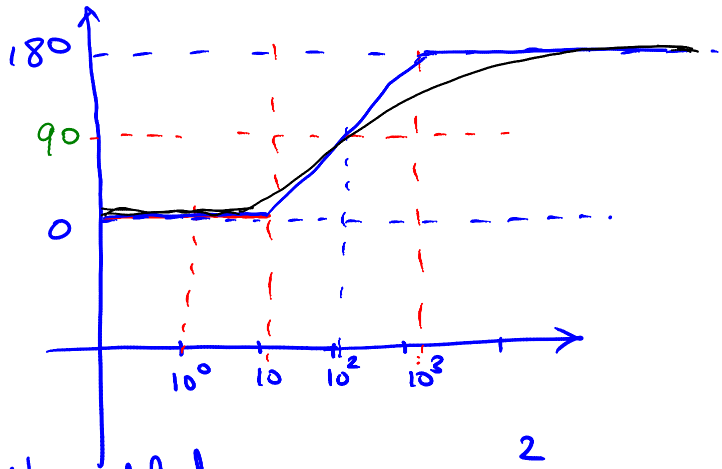
$\Rightarrow \quad G(j\omega) \approx \left(\frac{j\omega}{z}\right)^2 = \frac{\omega^2 j^2}{z^2} = -\frac{\omega^2}{z^2}$

Repeated zero (phase)

Saturday, October 24, 2009
6:24 PM

$\Rightarrow \quad \underline{\angle G(j\omega)} = +180^\circ$

Thus the phase plot looks like



phase plot of $\left(\frac{s}{100}+1\right)^2$.

Bode Plot of a second order system

Saturday, October 24, 2009
9:49 PM

$$H(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \qquad 0 < \xi < 1$$

$$= \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2\xi\frac{s}{\omega_0} + 1}$$

[Note : again $H(0) = 1$]

$$H(j\omega) = \frac{1}{-\frac{\omega^2}{\omega_0^2} + j\,2\xi\frac{\omega}{\omega_0} + 1}$$

$$= \frac{1}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + j\left(2\xi\frac{\omega}{\omega_0}\right)}$$

$$= \frac{1 - \omega^2/\omega_0^2 - j\,2\xi\,\omega/\omega_0}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2}$$

Second Order Systems

Saturday, October 24, 2009
9:51 PM

$$\therefore \quad |H(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}}$$

$$20\lg_{10}|H(j\omega)| = -20\lg\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}$$

Case 1: $\omega << \omega_0$ $\quad (\omega/\omega_0) << 1.$

$$H(j\omega) = 1$$

$$\Rightarrow \quad 20\lg_{10}|H(j\omega)| = 0 \quad dB$$

$$\angle H(j\omega) = 0.$$

Approximations

Saturday, October 24, 2009
9:57 PM

Case 2 ; $\omega \gg \omega_0$

$$H(j\omega) \approx \frac{-\omega^2/\omega_0^2}{\frac{\omega^4}{\omega_0^4}}$$

$$\approx -\frac{\omega_0^2}{\omega^2}$$

$$\Rightarrow 20\lg|H(j\omega)| = 20\lg_{10}\frac{\omega_0^2}{\omega^2}$$

$$= 20\lg_{10}\left(\omega/\omega_0\right)^{-2}$$

$$= -40\lg_{10}\frac{\omega}{\omega_0}$$

$$\angle H(j\omega) = -180$$

Resonant frequency

Saturday, October 24, 2009
9:53 PM

Case 3: Note that $\omega \simeq \omega_0$

$$|H(j\omega)|^2 = \frac{1}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2}$$

let $\Delta(\omega) = \left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_0}\right)^2$

If $\Delta(\omega)$ has a minimum at ω_r

then

$$\frac{d\Delta(\omega)}{d\omega} = 0$$

$$\Rightarrow 2\left(1-\frac{\omega^2}{\omega_0^2}\right)\left(-\frac{2\omega}{\omega_0^2}\right) + \underbrace{2\left(2\zeta\frac{\omega}{\omega_0}\right)\cdot\frac{2\zeta}{\omega_0}}_{4\frac{\omega}{\omega_0}\zeta\left(\frac{2\zeta}{\omega_0}\right)} = 0$$

$$\Rightarrow -\frac{4\omega}{\omega_0^2}\left[1-\frac{\omega^2}{\omega_0^2} - 2\zeta^2\right] = 0$$

$$\Rightarrow \omega_0^2 - \omega^2 - 2\zeta^2\omega_0^2 = 0$$

$$\Rightarrow \omega^2 = \omega_0^2 - 2\zeta^2\omega_0^2$$

$$\Rightarrow \omega_r = \omega_0\sqrt{1-2\zeta^2}$$

Resonant frequency and peak value

Saturday, October 24, 2009
9:54 PM

Note that the minimum exists only if

$$0 < \zeta < \frac{1}{\sqrt{2}} = 0.7$$

Thus there is a peak $\omega_r = \omega_0\sqrt{1-2\zeta^2}$
for $0 < \zeta < 0.7$

Peak Value:

Note that

$$|H(j\omega)|^2 = \frac{1}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\zeta\omega/\omega_0\right)^2}$$

at $\omega = \omega_r$; we have $\frac{\omega_r}{\omega_0} = \sqrt{1-2\zeta^2}$

$$\Rightarrow \left(\frac{\omega_r}{\omega_0}\right)^2 = 1-2\zeta^2$$

$$\Rightarrow \left(1-\frac{\omega_r^2}{\omega_0^2}\right)^2 + \left(2\zeta\omega_r/\omega_0\right)^2 = \left(1-(1-2\zeta^2)\right)^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2$$

$$\Rightarrow \quad = 4\zeta^4 + 4\zeta^2(1-2\zeta^2)$$

$$= 4\zeta^4 + 4\zeta^2 - 8\zeta^4$$

$$= 4\zeta^2 - 4\zeta^4 = 4\zeta^2(1-\zeta^2)$$

$$\therefore \quad |H(j\omega_r)|^2 = \frac{1}{4\zeta^2(1-\zeta^2)}$$

Peak Value

Saturday, October 31, 2009
10:56 PM

Thus

$$\boxed{|H(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

Underdamped, damped, overdamped Systems

Saturday, October 24, 2009
9:58 PM

- when $0 < \zeta < 0.7$ we say system is underdamped
- when $\zeta > 0.7$ we say system is overdamped.
- when $\zeta = 0.7$ we say system is critically damped.

As $\zeta^2 << 1$; $\omega_r \approx \omega_o$ is a reasonable approximation in most cases.

At $\omega = \omega_o$ the

$$\angle H(j\omega_o) = 90^\circ.$$

Bode Plot of a underdamped System.

Saturday, October 24, 2009
10:00 PM