Feedback

Single Input and Single output Interconnections

\[
\begin{align*}
    &Y \\
    \rightarrow & e_2 \rightarrow K \rightarrow u + d_1 \rightarrow e_1 \rightarrow G_1 \rightarrow y \\
    \downarrow & y_m \rightarrow + n \rightarrow + d_0
\end{align*}
\]

**Inputs:**

- \( r(t) \) is the reference input. A typical requirement is that the error \( e = r - y \) is small.
- \( d_1(t) \) is the input disturbance to the plant \( G_1 \).
- \( n \) is the noise that models the measurement noise (of measuring \( y \); the output of the plant \( G \)).
- \( d_0 \) is the output disturbance.

**Internal variables**

\( e_2, u, e_1, y, y_m \)
A transfer function from any particular input to any particular output can be found by setting all other inputs to zero.

Transfer function with input $r$

We set $d_n = n = d_0 = 0$

Transfer function from $r$ to $e_2$.

It follows that

\[ e_2 = r - y \]

and

\[ y = G_1 e_2 \]

\[ \Rightarrow e_2 = r - G_1 e_2 \]

\[ \Rightarrow (1 + G_1) e_2 = r \]

\[ \Rightarrow \frac{e_2}{y} = \frac{1}{1 + G_1} \]

Transfer function from $r$ to $u$.

\[ u = K e_2 = K \frac{1}{1 + G_1} r \]

\[ \Rightarrow \frac{u}{y} = \frac{K}{1 + G_1} \]
(c) Transfer function from $r \to e_1$

as $d_1 = 0$; $u = e_1$ and

$$
\frac{e_1}{r} = \frac{k}{1+gK}.
$$

(d) Transfer function from $r \to y$

$$
y = G_iK e_2
\Rightarrow y = G K \frac{1}{s}
\Rightarrow \frac{y}{y} = \frac{G K}{1+gK}
$$

(e) Transfer function from $r \to j m$

with $n = d_0 = 0$; $j m = y$

and

$$
\frac{j m}{y} = \frac{G K}{1+gK}.
$$
Transfer function with input $d_i$

Set $x = n = d_0 = 0$

(a) Output $e_1$:

$$e_1 = d_i + u$$

$$= d_i + Ke_2$$

$$= d_i + K(-y)$$

$$= d_i - KG e_1$$

$$\Rightarrow \frac{e_1}{d_i} = \frac{1}{1+KG}$$

(b) Output $y$ and $y_m$

$$\frac{y}{d_i} = \frac{y_m}{d_i} = \frac{G_i}{1+KG}$$

(c) Output $e_2$

$$e_2 = -y_m = -\frac{G_i}{1+KG}$$

(d) Output $u$

$$u = Ke_2 = -\frac{KG}{1+KG}$$
\[ \frac{\varphi_i}{\tau_i} = -\frac{\varphi_i}{1 + \kappa_0}. \]
Transfer functions with input $u_1$.

Set $r = d_1 - d_0 = 0$

1. **Output $e_2$**
   
   $$\frac{e_2}{n} = -\frac{1}{1+gK}$$

2. **Output $u$**
   
   $$\frac{u}{n} = -\frac{k}{1+gK}$$

3. **Output $y$**
   
   $$\frac{y}{n} = -\frac{gK}{1+gK}$$

4. **Output $y_m$**
   
   $$y_m = y + n = (1 - gK)n = \frac{1}{1+gK}$$
   
   $$y_m/n = Y/(1+gK).$$
Essentially, the complete set of transfer functions are:

1. \( \frac{1}{1+GK} \)
2. \( \frac{G}{1+GK} \)
3. \( \frac{K}{1+GK} \)
4. \( \frac{GK}{1+GK} \)

Thus, the above interconnection is bounded input and bounded output stable if and only if

\( \frac{1}{1+GK} \), \( \frac{G}{1+GK} \), \( \frac{K}{1+GK} \) and \( \frac{GK}{1+GK} \)

are all stable. I.e. the poles of the these transfer functions are in the whole left half plane.
Imp. closed-loop maps

Consider the following block diagram

\[ r \rightarrow e_2 \rightarrow k \rightarrow e_1 \rightarrow G \rightarrow \]

Figure 2:

Then

\[ e_2 = r - Ge_1 \]

\[ e_1 = d + ke_2 \]

\[ \Rightarrow e_1 = d + k(r - Ge_1) \]

\[ \Rightarrow e_1(1 + ka) = d + kr \]

\[ \Rightarrow e_1 = \frac{1}{1 + ka} \cdot d + \frac{k}{1 + ka} r \]

and

\[ e_2 = r - Ge_1 \]

\[ = r - \frac{Gd}{1 + ka} - \frac{ka}{1 + ka} r \]

\[ = \frac{1}{1 + ka} r - \frac{G}{1 + ka} d \]
Stability Equivalence

Transfer functions are
\[
\frac{1}{1+K_a}, \quad \frac{K}{1+K_a}, \quad \frac{G}{1+K_a}
\]

And thus the above interconnection is stable if
\[
\frac{1}{1+K_a}, \quad \frac{K}{1+K_a} \quad \text{and} \quad \frac{G}{1+K_a}
\]

all the poles in the strict left half plane.

Note that
\[
\frac{GK}{1+K_a} = 1 - \frac{1}{1+K_a}
\]

and thus, if \( \frac{1}{1+K_a} \) is stable then
\[
\frac{GK}{1+K_a}
\]

So is \( \frac{GK}{1+K_a} \).

Thus, the interconnection in Figure 1 is bounded input bounded output stable if and only if Figure 2 is bounded input bounded output stable (with inputs \( r \) and \( d \) and output \( e_1 \) and \( e_2 \)).
Thus we will focus on

\[ r \rightarrow o \rightarrow k \rightarrow o \rightarrow g \]

Thus we have

**Theorem 1:**

Bounded input bounded stability if and only if

\[ \frac{G}{1+GK}, \frac{K}{1+GK} \text{ and } \frac{1}{1+GK} \]

are stable transfer functions.

2) Let \( G = \frac{P_a}{\partial a} \) and \( K = \frac{P_k}{\partial k} \), where \( P_a, d_a, P_k, d_k \) are all polynomials.

(We assume that \( P_a \) and \( d_a \) have no common roots and we assume \( P_k \) and \( d_k \) have no common roots.)
Note that assuming that $\mathcal{N}_d, \mathcal{M}_d$ have no common roots and $\mathcal{N}_e, \mathcal{M}_e$ have no common roots does not imply $\mathcal{N}_d \mathcal{N}_e$ and $\mathcal{D}_d \mathcal{D}_e$ have no common roots.

\textbf{Example:} \quad \text{let} \quad K = \frac{(s-1)}{(s+3)(s+2)} \quad \text{and} \quad G = \frac{(s+3)}{(s+2)(s-1)}

\begin{align*}
\mathcal{N}_K &= (s-1) \quad \mathcal{D}_K &= (s+3)(s+2) \\
\mathcal{N}_G &= (s+3) \quad \mathcal{D}_G &= (s+2)(s-1)
\end{align*}

and

\begin{align*}
\mathcal{N}_d \mathcal{N}_e &= (s-1)(s+3) \\
\mathcal{D}_d \mathcal{D}_e &= (s+3)(s+2)(s-1)
\end{align*}

Thus, $\mathcal{D}_d \mathcal{D}_e$ and $\mathcal{N}_d \mathcal{N}_e$ have two common terms $(s+3)$ and $(s-1)$. 
Theorem 2:
The interconnection shown in Figure 1 and Figure 2 are bounded input bounded output stable if the polynomial $\frac{dudk + nk}{G}$ has all roots in the strict left half plane.

Proof: Note that from the previous theorem, Figure 1 and Figure 2 are BIBO stable if and only if

$$\frac{1}{1+6k} = \frac{1}{1+nk/d_k} = \frac{d_k}{d_k+nk}$$

$$\frac{G}{1+6k} = \frac{n_k/d_k}{d_k+nk}$$

$$\frac{K}{1+6k} = \frac{n_k/d_k}{d_k+nk}$$

have all poles in the strict left half plane.

Clearly if $\frac{dudk + nk}{G}$ has no roots in the strict left half plane then...
all the above transfer functions will have no poles in the right half plane and BIBO stability follows.

\[ I \]

Remark: Note that it is possible that \( \frac{1}{1+uK} \) is stable without \( \frac{G}{1+uK} \) or \( \frac{K}{1+uK} \) being stable.

Example: Let \( G = \frac{1}{s-1} \) and \( K = \frac{s-1}{s+2} \)

Thus, \( 1+uK = 1 + \frac{1}{s-1} \frac{s-1}{s+2} \)

\[ = 1 + \frac{1}{s+2} \]
\[ = \frac{s+3}{s+2} \]

\( \Rightarrow \frac{1}{1+uK} = \frac{s+2}{s+3} \) which is stable

However \( \frac{G}{1+uK} = \frac{1}{(s-1) \frac{s+2}{s+3}} \) which is not stable.
Pole zero Cancellation

The problem is unstable pole-zero cancellation between \( G \) and \( K \).

Note that \( G \) has an unstable pole at 1 and \( K \) has an unstable zero at the same location 1. Thus

\[
GK = \left( \frac{1}{s-1} \right) \left( \frac{s+1}{s+2} \right) = \frac{1}{s+2}
\]

and there is a cancellation of an unstable factor.

The following theorem holds:

**Theorem 3:** The interconnection in Figure 1 and Figure 2 are BIBO stable if and only if

- There are no unstable pole-zero cancellations when forming the product \( GK \)

- \( 1+GH \) has no zeros in the right half plane.
Proof: Note that with $G = \frac{N_k}{\frac{d_k}{d_k}}$; $K = \frac{N_k}{d_k}$

then $1 + GK = \frac{N_0N_k + \frac{d_k}{d_k}}{\frac{d_k}{d_k}}$.

Q Suppose Q and R hold.

Note that if $N_0N_k + \frac{d_k}{d_k}$ has all roots in the strict left half plane the the system is BIBO stable (Thorem 2).

As Q holds $(1 + Gk)$ has no zeros in the right half plane.

Thus, $N_0N_k + \frac{d_k}{d_k}$ has no zeros in the right half plane.

Thus, $N_0N_k + \frac{d_k}{d_k}$ can have zero in the right half plane only if such a factor is cancelled by * the same factor of $\frac{d_k}{d_k}$, i.e. there is a 0 in the right half plane with $(N_0N_k + \frac{d_k}{d_k}) (z_o) = 0$

$(\frac{d_k}{d_k}) (z_o) = 0$
but this implies that
\[ 0 = (\nu_1\nu_k)(s_0) + (d_k)(s_0) \]
\[ = (\nu_1\nu_k)(s_0) + 0 \]
\[ \Rightarrow (\nu_1\nu_k)(s_0) = 0 \]

Thus, \( s_0 \) is such that \( (\nu_1\nu_k)(s_0) = (d_k)(s_0) \)

and thus, there is a common factor \( (s-s_0) \) between \( \nu_1\nu_k \) and \( d_k \). Thus, there is a questionable pole-zero cancellation when the product \( G_1 = \nu_1\nu_k \) is formed.

and this is not allowed by (b).

Thus, if conditions (1) and (b) are met then the system in Figure 1 and (2) are BIBO stable.
Summary:

For BIBO stability of Figure 1 ascertain that

a) There is no unstable pole-zero cancellation in forming the product $GK$

b) $GK$ has no zeros in the right half-plane.

Equivalently ascertain that

Routh Hurwitz and see if polynomial $du+u$ has any roots in the right half-plane.