# Linear Programming

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The standard Linear Programming (SLP) problem:

$$\underset{x \in R^n}{\text{minimize}} \quad \overbrace{c_1x_1 + c_2x_2 + \ldots + c_nx_n}^{c^Tx}$$

subject to

$$\overbrace{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}}^{Ax=b}$$

$x_i \geq 0$ for all $i = 1, \ldots, n$

Define the feasible set of the SLP as

$$\Lambda := \{x \in R^n | Ax = b,\ x \geq 0\}.$$

The SLP is given by

$$\text{minimize } \{c^T x | x \in \Lambda\}.$$

**Theorem 1.** *Consider the following problems*

$$\mu = \min\{\tilde{c}^T z | A_1 z \leq b_1,\ A_2 z = b_2 \text{ and } z \geq 0\} \tag{1}$$

*and*

$$\nu = \min\{c^T x | Ax = b, x \geq 0\} \tag{2}$$

*where*

$$c = \begin{bmatrix} \tilde{c} \\ 0 \end{bmatrix}, A = \begin{bmatrix} A_1 & I \\ A_2 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

*Then*

$$\mu = \nu$$

*If the optimal solution of (1) is $z^o$ then an optimal solution $x^o$ of (2) is given by*

$$x^o = \begin{bmatrix} z^0 \\ y^o \end{bmatrix}$$

*where $y^o \geq 0$ and vice versa.*

**Proof:** Note that as $x^o$ is an optimal solution of (2) it follows that

$$\nu = c^T x^o,\ Ax^o = b \text{ and } x^o \geq 0.$$

Partition $x^o$ appropriately as

$$x^o = \begin{bmatrix} z^1 \\ y^o \end{bmatrix}$$

where $z$ has the same dimension as $\tilde{c}$. Then it follows that

$$z^1 \geq 0,\ A_1 z^1 + y^o = b_1,\ y^o \geq 0 \text{ and } A_2 z^1 = b_2.$$

This implies that

$$z^1 \geq 0,\ A_1 z^1 \leq b_1,\ A_2 z^1 = b_2$$

and thus $z^1$ is a feasible element for the optimization problem of (1). Thus it follows that

$$\nu = c^T x^o = \tilde{c}^T z^1 \geq \min\{\tilde{c}^T z | A_1 z \leq b_1,\ A_2 z = b_2 \text{ and } z \geq 0\} = \mu.$$

Note that as $z^o$ is an optimal solution of (1) it follows that

$$\mu = \tilde{c}^T z^o,\ A_1 z^o \leq b_1,\ A_2 z^o = b_2 \text{ and } z^o \geq 0.$$

Define

$$y^1 := b_1 - A_1 z^o \geq 0.$$

Define

$$x^1 = \begin{bmatrix} z^o \\ y^1 \end{bmatrix}.$$

Then it follows that

$$x^1 \geq 0,\ Ax^1 = b \text{ and } x^1 \geq 0$$

and thus $x^1$ is a feasible element for the optimization problem of (2). Thus it follows that

$$\mu = \tilde{c}^T z^o = c^T x^1 \geq \min\{c^T x^1 | Ax = b \text{ and } x \geq 0\} = \nu = \tilde{c}^T z^1 \geq \mu.$$

This proves $\mu = \nu$. Also we have shown that if the optimal solution of (1) is $z^o$ then an optimal solution $x^o$ of (2) is given by

$$x^o = \begin{bmatrix} z^0 \\ y^o \end{bmatrix}$$

where $y^o \geq 0$ and vice versa. ∎

**Theorem 2.** *Consider the following problems*

$$\mu = \min\{\begin{pmatrix} \tilde{c}_1^T & \tilde{c}_2^T \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix} | A_1 z + A_2 y = b,\ z \geq 0\} \tag{3}$$

*and*

$$\nu = \min\{c^T x | Ax = b, x \geq 0\} \tag{4}$$

*where*

$$c = \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ -\tilde{c}_2 \end{bmatrix}, A = \begin{bmatrix} A_1 & A_2 & -A_2 \end{bmatrix}.$$

*Then*

$$\mu = \nu$$

*If the optimal solution of (3) is* $\begin{pmatrix} z^o \\ y^o \end{pmatrix}$ *then an optimal solution* $x^o$ *of (4) is* *given by*

$$x^o = \begin{bmatrix} z^0 \\ u^o \\ v^o \end{bmatrix}$$

*where* $y^o = u^o - v^o \geq 0$ *and vice versa.*

**Proof:** Note that as $x^o$ is an optimal solution of (4) it follows that

$$\nu = c^T x^o,\ Ax^o = b \text{ and } x^o \geq 0.$$

Partition $x^o$ appropriately as

$$x^o = \begin{bmatrix} z^1 \\ u^o \\ v^o \end{bmatrix}$$

where $z^1,\ u^o$ and $v^o$ have dimensions same as $\tilde{c}_1,\ \tilde{c}_2$ and $\tilde{c}_2$ respectively. Then it follows that

$$z^1 \geq 0,\ A_1 z^1 + A_2(u^0 - v^o) = b,\ u^o,\ v^o \geq 0.$$

Let $y^1 := u^0 - v^o$. This implies that

$$z^1 \geq 0,\ A_1 z^1 + A_2 y^1 = b$$

and thus $\begin{pmatrix} z^1 \\ y^1 \end{pmatrix}$ is a feasible element for the optimization problem of (3).
Thus it follows that

$$\begin{array}{rcl}
\nu & = & c^T x^o \\
 & = & \begin{pmatrix} \tilde{c}_1^T & \tilde{c}_2^T & -\tilde{c}_2^T \end{pmatrix} \begin{pmatrix} z^1 \\ u^o \\ v^o \end{pmatrix} = \tilde{c}_1^T z^1 + \tilde{c}_2^T (u^o - v^o) \\
 & = & \begin{pmatrix} \tilde{c}_1^T & \tilde{c}_2^T \end{pmatrix} \begin{pmatrix} z^1 \\ y^1 \end{pmatrix} \\
 & \geq & \min\{ \begin{pmatrix} \tilde{c}_1^T & \tilde{c}_2^T \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix} | A_1 z + A_2 y = b,\ z \geq 0\} \\
 & = & \mu
\end{array}$$

Note that as $\begin{pmatrix} z^o \\ y^o \end{pmatrix}$ is an optimal solution of (3) it follows that

$$\mu = \tilde{c}_1^T z^o + \tilde{c}_1^T y^o, \; A_1 z^o + A_2 y^o = b, \; z^o \geq 0.$$

Define $u^o$ and $v^o$ to satisfy

$$\begin{array}{rcr} u^1(i) & := & y^o(i) \text{ if } y^o(i) \geq 0 \\ & = & 0 \text{ if } y^o(i) < 0 \\ \\ v^1(i) & = & 0 \text{ if } y^o(i) \geq 0 \\ & = & -y^o(i) \text{ if } y^o(i) < 0. \end{array}$$

for all $i = 1, \ldots, n_y$ $n_y$ being the dimension of $y$. Note that

$$\begin{array}{rcl} u^1 & \geq & 0 \\ v^1 & \geq & 0 \text{ and} \\ y^o & = & u^1 - v^1. \end{array}$$

Therefore it follow that

$$\begin{aligned} A_1 z^o + A_2 u^1 - A_2 v^1 = A_1 z^o + A_2 y^o &= b \\ u^1 \geq 1 & \\ v^1 \geq 0 & \end{aligned}$$

Thus $x^1 := \begin{pmatrix} z^o \\ u^1 \\ v^1 \end{pmatrix}$ is a feasible solution for (4). Thus it follows that

$$\begin{aligned} \mu &= \tilde{c}_1^T z^o + \tilde{c}_2^T y^o = \tilde{c}_1^T z^o + \tilde{c}_2^T u^o - \tilde{c}_2^T v^o \\ &= c^T x^1 \\ &\geq \min\{c^T x | Ax = b \text{ and } x \geq 0\} \\ &= \nu \\ &= \tilde{c}_1^T z^1 + \tilde{c}_2^T (u^o - v^o) \\ &\geq \mu \end{aligned}$$

This proves $\mu = \nu$. Also we have shown that if the optimal solution of (3) is

$\begin{pmatrix} z^o \\ y^o \end{pmatrix}$ then an optimal solution $x^o$ of (4) is given by

$$x^o = \begin{bmatrix} z^0 \\ u^o \\ v^o \end{bmatrix}$$

where $y^o = u^o - v^o \geq 0$ and vice versa. ∎

# Feasible solution and Optimal solution

**Definition 1.** *Consider the Standard Linear Programming (SLP) problem*

$$\mu = \min\{c^T x | Ax = b,\ x \geq 0,\ x \in R^n\}$$

*where $A$ is a $m \times n$ matrix. Any $x \in R^n$ that satisfies $Ax = b,\ x \geq 0$ is a feasible solution. If $x^o$ is such that*

$$\mu = c^T x^o,\ Ax^o = b \text{ and } x \geq 0,$$

*then $x^o$ is an optimal solution.*

# Basic Solution, basic variable and nonbasic variables

**Definition 2.** *Consider the Standard Linear Programming (SLP) problem*

$$\min\{c^T x | Ax = b,\ x \geq 0,\ x \in R^n\}$$

*where* $A$ *is a* $m \times n$ *matrix. Suppose*

$$Rank(A) = m.$$

*Suppose*

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

*is such that only $m$ elements $\{x_{k_1}, x_{k_2}, \ldots, x_{k_m}\}$ are non zero with*

$$\begin{pmatrix} x_{k_1} \\ x_{k_2} \\ \vdots \\ x_{k_m} \end{pmatrix} = B^{-1}b, \; B = \begin{bmatrix} a_{k_1} & a_{k_2} & \ldots & a_{k_m} \end{bmatrix}.$$

*Then $x$ is a basic solution of the SLP.*

*The variables $\{x_{k_1}, x_{k_2}, \ldots, x_{k_m}\}$ are called the basic variables associated with the matrix $B$. The variables $x_i$ with $i \notin \{k_1, k_2, \ldots, k_m\}$ are called the non-basic variables.*

Note that

- $B$ is a matrix formed by $m$ linearly independent columns of $A$.
- Basic solution depends only on $A$ and $b$ and not on $c$.

- A non-basic variable is set to zero in a basic solution
- A basic variable can be zero in a basic solution.
- There are only finitely many basic solutions associated with $A \in R^{m \times n}$ and $b \in R^m$.

**Definition 3.** *A basic solution is said to be degenerate if any of the basic variables is zero.*

**Definition 4.** $x$ *is said to be basic feasible solution if* $x$ *is basic and is feasible.*

**Definition 5.** $x$ *is said to be basic optimal solution if* $x$ *is basic and is optimal.*

# The Fundamental Theorem of Linear Programming

**Theorem 3.** *Consider the optimization problem*

$$\min\{c^T x | Ax = b,\ x \geq 0,\ x \in R^n\},$$

*where* $A \in R^{m \times n}$ *has rank* $m$. *Then*

1. *If there exists a feasible solution then there exists a basic feasible solution.*

2. *If there is an optimal solution then there is a basic optimal solution*

- SLP has only finitely many basic solutions.

- Fundamental theorem on Linear Programming asserts that LP can be solved in a finite number of steps

**Proof of (1.):** Let $x \in R^n$ be a feasible solution. Suppose only $p$ elements of the vector $x$ be nonzero. Without loss of generality assume these variables to be $x_1, \ldots, x_p$. Thus

$$x_{p+1} = x_{p+2} = \ldots = x_n = 0.$$

Note also that $Ax = b$ and $x \geq 0$. Let

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix},$$

where $a_i$ denotes the $i^{th}$ column of $A$. Then as $Ax = b$ we have

$$\sum_{i=1}^{n} a_i x_i = b \Rightarrow \sum_{i=1}^{p} a_i x_i = b.$$

Case1: Suppose $a_1, \ldots, a_p$ are independent set of vectors. Then as $p \leq m$ as $Rank(A) = m$. One can add columns $a_{i_{p+1}}, \ldots, a_{i_m}$ such that the

$$B = \begin{bmatrix} a_1 & \cdots & a_p & a_{i_{p+1}} & a_{i_{p+2}} & \cdots & a_{i_m} \end{bmatrix},$$

has independent columns and thus is invertible. It is evident that $x \geq 0,\ Ax = b$ and the nonzero variables are basic variables associated with the matrix $B$ above. Thus it follows that $x$ is a basic feasible solution.

Case 2: Suppose the columns $a_1, \ldots, a_p$ form a dependent set. Then there exists real variables $y_1, \ldots, y_p$ with at least one element strictly positive such that

$$\sum_{i=1}^{p} y_i a_i = 0.$$

Let

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}.$$

It is evident that $Ay = 0$. Let

$$\epsilon := \min\{\frac{x_1}{y_1}, \ldots, \frac{x_p}{y_p} | y_i > 0\} > 0.$$

Let

$$z = x - \epsilon y.$$

Then

$$z \geq 0,\ Az = A(x - \epsilon y) = Ax - \epsilon Ay = Ax = b$$

and $z$ has $p-1$ nonzero elements. Thus $z$ is a feasible solution and has at most $p-1$ nonzero elements. This process can be continued to a stage when the non-zero elements of a feasible element are associated with independent columns and then we revert to Case 1.

This proves the first part of the theorem.

**Proof of (2.):** Suppose $\tilde{x}$ is such that

$$\mu = c^T\tilde{x},\ A\tilde{x} = b \text{ and } \tilde{x} \geq 0$$

that is $\tilde{x}$ is an optimal solution. Suppose only $p$ elements of the vector $\tilde{x}$ be nonzero. Without loss of generality assume these variables to be $x_1, \ldots, x_p$. Thus

$$\tilde{x}_{p+1} = \tilde{x}_{p+2} = \ldots = \tilde{x}_n = 0.$$

Note also that $A\tilde{x} = b$ and $\tilde{x} \geq 0$. Let

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix},$$

where $a_i$ denotes the $i^{th}$ column of $A$. Then as $A\tilde{x} = b$ we have

$$\sum_{i=1}^{n} a_i\tilde{x}_i = b \Rightarrow \sum_{i=1}^{p} a_i\tilde{x}_i = b.$$

Case1: Suppose $a_1, \ldots, a_p$ are independent set of vectors. Then as $p \leq m$ as $Rank(A) = m$. Using results from linear algebra, one can add columns $a_{i_{p+1}}, \ldots, a_{i_m}$ such that the

$$B = \left[ \begin{array}{cccccccc} a_1 & \cdots & a_p & a_{i_{p+1}} & a_{i_{p+2}} & \cdots & a_{i_m} \end{array} \right],$$

has independent columns and thus is invertible. It is evident that $\tilde{x} \geq 0, \; A\tilde{x} = b$ and the nonzero variables are basic variables associated with the matrix $B$ above. Thus it follows that $\tilde{x}$ is a basic feasible solution. $\tilde{x}$ is an optimal solution too and thus $\tilde{x}$ is a basic optimal solution.

Case 2: Suppose the columns $a_1, \ldots, a_p$ forms a dependent set. Then there exists real variables $y_1, \ldots, y_p$ with at least one element strictly positive such that

$$\sum_{i=1}^{p} y_i a_i = 0.$$

Let

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} .$$

It is evident that $Ay = 0$.

Let

$$\delta := \min\{\frac{\tilde{x}_1}{|y_1|}, \ldots, \frac{\tilde{x}_p}{|y_p|} | y_i \neq 0\} > 0.$$

Let $\tilde{\epsilon}$ be any real number such that

$$|\tilde{\epsilon}| < \delta.$$

Then

$$\tilde{x} - \tilde{\epsilon} y \geq 0 \text{ and } A(\tilde{x} - \tilde{\epsilon} y) = b.$$

Thus $\tilde{x} - \tilde{\epsilon}y$ is a feasible solution. Suppose $c^T y \neq 0$ then we can choose $\tilde{\epsilon}$ such that $0 < |\tilde{\epsilon}| \leq \delta$ and $sgn(\tilde{\epsilon}) = sgn(c^T y)$. Then

$$c^T(\tilde{x} - \tilde{\epsilon}y) = c^T\tilde{x} - \tilde{\epsilon}c^T y = c^T\tilde{x} - c^T - |\tilde{\epsilon}c^T y| < c^T\tilde{x}.$$

As $(\tilde{x} - \tilde{\epsilon}y)$ is a feasible solution $\tilde{x}$ cannot be an optimal solution. This a contradiction and thus

$$c^T y = 0.$$

Now let

$$\epsilon := \min\{\frac{x_1}{y_1}, \ldots, \frac{x_p}{y_p} | y_i > 0\} > 0.$$

Let

$$z = x - \epsilon y.$$

Then $z$ is a feasible solution with

$$c^T z = c^T(\tilde{x} - \epsilon c^T y) = c^T\tilde{x} = \mu$$

and thus $z$ is an optmal solution. Also $z$ has at most $p-1$ nonzero elements. This process can be continued to a stage when the only non-zero terms in the optimal solution are associated with independent columns of $A$.

This proves $(2.)$.

∎

**Definition 6. [Convex sets]** *A subset $\Omega$ of a vector space $X$ is said to be convex if for any two elements $c_1$ and $c_2$ in $\Omega$ and for a real number $\lambda$ with $0 < \lambda < 1$ the element $\lambda c_1 + (1-\lambda)c_2 \in \Omega$ (see Figure* ***??****). The set $\{\}$ is assumed to be convex.*

**Theorem 4.** *Let $\Lambda_\alpha,\ \alpha \in \mathcal{S}$ be an arbitrary collection of convex sets. Then*

$$\bigcap_{\alpha \in \mathcal{S}} \Lambda_\alpha$$

*is a convex set.*

**Theorem 5.** *Suppose $K$ and $G$ are convex subsets of a vector space $X$. Then*

$$K + G := \{x \in X | x = x_k + x_G,\ x_K \in K \text{ and } x_G \in G\}$$

*is convex.*

**Definition 7.** *Let $S$ be an arbitrary set of a vector space $X$. Then the convex hull of $S$ is the smallest convex set containing $S$ and is denoted by $co(S)$.*

Note that

$$co(S) = \bigcap \Lambda_\alpha$$

where $\Lambda_\alpha$ is any set that contains $S$.

**Definition 8. [Convex combination]** *A vector of the form $\sum_{k=1}^{n} \lambda_k x_k$, where $\sum_{k=1}^{n} \lambda_k = 1$ and $\lambda_k \geq 0$ for all $k = 1, \ldots, n$ is a convex combination of the vectors $x_1, \ldots, x_n$.*

**Definition 9. [Cones]** *A subset $C$ of a vector space $X$ is a cone if for every non-negative $\alpha$ in $R$ and $c$ in $C, \alpha c \in C$.*

*A subset $C$ of a vector space is a convex cone if $C$ is convex and is also a cone.*

**Definition 10. [Positive cones]** *A convex cone $P$ in a vector space $X$ is a positive convex cone if a relation $' \geq '$ is defined on $X$ based on $P$ such that for elements $x$ and $y$ in $X, x \geq y$ if $x - y \in P$. We write $x > 0$ if $x \in int(P)$. Similarly $x \leq y$ if $x - y \in -P := N$ and $x < 0$ if $x \in int(N)$.*

**Example 1.** *Consider the real number system $R$. The set*

$$P := \{x : x \text{ is nonnegative}\},$$

*defines a cone in $R$. It also induces a relation $\geq$ on $R$ where for any two elements $x$ and $y$ in $R, x \geq y$ if and only if $x - y \in P$. The convex cone $P$ with the relation $\geq$ defines a positive cone on $R$.*

**Definition 11. [Convex maps]** *Let X be a vector space and Z be a vector space with positive cone P. A mapping,* $G : X \to Z$ *is convex if* $G(tx + (1-t)y) \leq tG(x) + (1-t)G(y)$ *for all* $x, y$ *in* $X$ *and* $t$ *with* $0 \leq t \leq 1$ *and is strictly convex if* $G(tx + (1-t)y) < tG(x) + (1-t)G(y)$ *for all* $x \neq y$ *in* $X$ *and* $t$ *with* $0 < t < 1$.

**Definition 12. [Extreme points]** *Let* $C$ *be a convex set. Then* $a \in C$ *is said to be an extreme point of the set* $C$ *if for any* $x,\ y \in C$ *and* $0 < \lambda < 1$

$$\lambda x + (1-\lambda)y = a$$

*implies that*

$$x = y = a.$$

Note that the feasible set of a SLP is given by

$$\Lambda = \{x \in R^n | Ax = bx \geq 0\}.$$

Clearly if $x$ and $y \in \Lambda$ then it follows that

$A(\lambda x+(1-\lambda)y) = \lambda Ax+(1-\lambda)Ay = \lambda b+(1-\lambda)Ab = b$ and $(\lambda x+(1-\lambda)y) \geq 0.$

Thus $(\lambda x + (1 - \lambda)y) \in \Lambda$ if $x$ and $y \in \Lambda$. Thus $\Lambda$ is convex.

# Equivalence of extreme points and basic solutions

**Theorem 6.** *Let $A$ be a $m \times n$ matrix with $Rank(A) = m$ and let*

$$\Lambda = \{x \in R^n | Ax = bx \geq 0\}.$$

*Then a vector $x$ is an extreme point of $\Lambda$ if and only if $x$ is a basic feasible solution.*

**Proof:** Suppose $x$ is a basic feasible solution. Assume without loss of generality that the basic variables are the first $m$ elements of $x$ given by $x_i, \ i = 1, \ldots, m$. Also let

$$B = \left[ \begin{array}{cccc} a_1 & a_2 & \cdots & a_m \end{array} \right].$$

Then it follows that

$$x_1 a_1 + x_2 a_2 + \ldots + x_m a_m = b$$

or in other words

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = B^{-1}b.$$

Now suppose $0 < \lambda < 1$ and $y$ and $z \in \Lambda$ are such that

$$\lambda y + (1-\lambda)z = x.$$

Thus it follows that

$$\lambda y_i + (1-\lambda)z_i = x_i \text{ for all } i = 1, \ldots, n.$$

Note that as $x_i = 0$ for all $i = m+1, \ldots, n,\ \lambda > 0$ and $(1-\lambda) > 0$ it follows that

$$y_i = z_i = 0 \text{ for all } i = m+1, \ldots, n.$$

Note that as $y$ and $z \in \Lambda$, $Ay = Az = b$ and thus

$$y_1 a_1 + \ldots + y_m a_m = z_1 a_1 + \ldots + z_m a_m = b.$$

Thus

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} = B^{-1} b = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

Thus

$$y = z = x.$$

Thus we have shown that every basic feasible solution is an extreme point.

Suppose $x$ is an extreme point of the set $\Lambda$. Without loss of generality assume that $x_1, \ldots, x_p$ are the only non-zero elements of $x$. Suppose $a_1, \ldots, a_p$ form a dependent set. Then there exists scalars $y_1, \ldots, y_p$ not all zero such that

$$y_1 a_1 + \ldots + y_p a_p = 0.$$

Let

$$\epsilon = \min\{x_i/|y_i||i \in \{1,\ldots,p\} \text{ and } y_i \not= 0\} > 0.$$

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \\ 0 \\ \vdots 0 \end{bmatrix}_{n\times 1} .$$

Note that

$$x - \epsilon y \geq 0,\ x + \epsilon y \geq 0,\ A(x - \epsilon y) = b \text{ and } A(x + \epsilon y) = b.$$

Also,

$$x = \frac{1}{2}(x - \epsilon y) + \frac{1}{2}(x + \epsilon y).$$

Clearly

$$x \neq x - \epsilon \text{ and } x \neq (x + \epsilon y)$$

as $y \neq 0$ and $\epsilon \neq 0$. Thus we have written $x$ as a non trivial convex combination of $x - \epsilon$ and $(x + \epsilon y)$. Thus $x$ is not an extreme point. This is a contradiction and therefore $a_1, \ldots, a_p$ are independent. Thus $x$ is a basic feasible solution. ∎

The following corollaries follow easily from Theorem 3 and Theorem 6.

**Corollary 1.** *Suppose $Rank(A) = m$ where $A$ is a $m \times n$ matrix. The feasible set of the SLP*

$$\Lambda = \{x \in R^n | Ax = b \textit{ and } x \geq 0\}$$

*is nonempty if and only if there exists an extreme point of $\Lambda$.*

**Corollary 2.** *Suppose $Rank(A) = m$ where $A$ is a $m \times n$ matrix. Let the feasible set of the SLP be*

$$\Lambda = \{x \in R^n | Ax = b \textit{ and } x \geq 0\}.$$

*Then an optimal solution to the SLP exists if and only if an optimal solution to the SLP that is also an extreme point of $\Lambda$ exists.*

## Which basic variable becomes non-basic

Consider the SLP with

$$\Lambda = \{x \in R^n | Ax = b,\ x \geq 0\}$$

as the feasible set. Lets assume that

- the first $m$ columns of $A$ are independent
- the basic solution $x$ associated with the first $m$ columns is feasible i.e. $x \geq 0$.

Thus

$x_1 a_1 + x_2 a_2 + \ldots + x_m a_m = b,\ x_{m+1} = x_{m+2} = \ldots = 0$ and $x_i > 0$ for all $i = 1, \ldots, m$.

The columns $a_1, a_2 \dots a_m$ are the basic columns. Suppose it is determined that a nonbasic column $a_q$ should become a basic column. Then we have to determine which column has to leave from the basic set. Note that $a_1, a_2 \dots a_m$ form a basis for $R^m$. Therefore we can write

$$a_q = y_{1q}a_1 + \dots + y_{mq}a_m.$$

Suppose the variable associated with $a_q$ is increased from $0$ to $\epsilon > 0$ while keeping all other non-basic variables $0$. Then to maintain feasibility we have that coefficients of the initial basic set $a_1, \dots, a_m$ will be altered to satisfy

$$\bar{x}_1 a_1 + \bar{x}_2 a_2 + \dots + \bar{x}_m a_m + \epsilon a_q = b$$

where $\bar{x}_i,\ i = 1, \dots, m$ have to be determined. Clearly

$$\begin{aligned} \bar{x}_1 a_1 + \bar{x}_2 a_2 + \dots + \bar{x}_m a_m &= b - \epsilon a_q \\ &= \textstyle\sum_{i=1}^m x_i a_i - \epsilon \sum_{i=1}^m y_{iq} a_i \\ &= \textstyle\sum_{i=1}^m (x_i - \epsilon y_{iq}) a_i \end{aligned}$$

. This implies that

$$\sum_{i=1}^{m} [\bar{x} - (x_i - \epsilon y_{iq})] a_i = 0.$$

Thus

$$\bar{x}_i = x_i - \epsilon y_{iq} \text{ for all } i = 1, \ldots, m.$$

Thus we have

$$\sum_{i=1}^{m} [(x_i - \epsilon y_{iq})] a_i + \epsilon a_q = b.$$

Thus we have a solution to the equation $Az = b$ given by

$$\bar{x} = \begin{bmatrix} x_1 - \epsilon y_{1q} \\ \vdots \\ x_m - \epsilon y_{mq} \\ 0 \\ \vdots \\ 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the $\epsilon$ is the $q^{th}$ element.

# Feasibility of $\bar{x}$

Note that $A\bar{x} = b$ where

$$\bar{x} = \begin{bmatrix} x_1 - \epsilon y_{1q} \\ \vdots \\ x_m - \epsilon y_{mq} \\ 0 \\ \vdots \\ 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

To be feasible $\bar{x} \geq 0$. This may be satisfied by an appropriate choice of $\epsilon$. Indeed there are two possible case for feasibility of $\bar{x}$.

Case 1: $y_{iq} \leq 0$ for all $i = 1, \ldots, m$. In this case $\bar{x}$ is feasible (that is $\bar{x} \in \Lambda$) for any $\epsilon > 0$. Also we can conclude that $\Lambda$ is an unbounded set.

Case 2: there exists at least one $i_0 \in \{1, 2, \ldots, m\}$ such that $y_{i_0 q} > 0$. In this case for any $0 \leq \epsilon \leq \epsilon_M$, $\bar{x}$ is feasible with

$$\epsilon_M = \min_{i=1,\ldots,m} \{\frac{x_i}{y_{iq}} | y_{iq} > 0\} \geq 0.$$

# Making $\bar{x}$ basic feasible solution with $a_q$ basic

Note that

$$\bar{x} = \begin{bmatrix} x_1 - \epsilon y_{1q} \\ \vdots \\ x_m - \epsilon y_{mq} \\ 0 \\ \vdots \\ 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Case 1: If $y_{iq} \leq 0$ for all $i = 1, \ldots, m$ then it is not possible to make $x_i - \epsilon y_{iq} = 0$ any $\epsilon$ if $x_i > 0$. If the initial basic feasible solution was degenerate with $x_k = 0, \; k \in \{1, \ldots, m\}$ then by choosing $\epsilon = 0$ one can swap the role of $a_k$ and $a_q$. However in this case the solution $\bar{x} = x$.

Case 2: there exists at least one $i_0 \in \{1, 2, \ldots, m\}$ such that $y_{i_0 q} > 0$. In this case let $\epsilon = \epsilon_M$, with

$$\epsilon_M = \min_{i=1,\ldots,m} \{\frac{x_i}{y_{iq}} | y_{iq} > 0\} \geq 0.$$

Let $p$ a minimizing index above. Then $a_p$ can be replaced with $a_q$ in the basic column set. Note again that if $x_p = 0$ then again the new solution $\bar{x} = x$.

# Boundedness and Non-degeneracy assumption

$$\bar{x} = \begin{bmatrix} x_1 - \epsilon y_{1q} \\ \vdots \\ x_m - \epsilon y_{mq} \\ 0 \\ \vdots \\ 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If we make the assumption that $\Lambda$ is bounded and that $x$ is a non-degenerate basic feasible solution the following steps can be performed to determine which basic column leaves the basic column set to allow $a_q$ to enter the basic column set.

1.

$$\epsilon_M = \min_{i=1,\ldots,m} \{\frac{x_i}{y_{iq}} | y_{iq} > 0\} > 0$$

with

$$p = arg\{\min_{i=1,\ldots,m} \{\frac{x_i}{y_{iq}} | y_{iq} > 0\}\}.$$

2. Let

$$\bar{x} = \begin{bmatrix} x_1 - \epsilon_M y_{1q} \\ \vdots \\ x_m - \epsilon_M y_{mq} \\ 0 \\ \vdots \\ 0 \\ \epsilon_M \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Note that $\bar{x} \geq 0$ and it has the $p^{th}$ element zero. The only possible nonzero elements are belong to the set $\{1, \ldots, (p-1), (p+1), \ldots, m\} \cup \{q\}$.

# Effect on the Cost of changing a basic feasible solution

Suppose $x_{bfs}$ is a basic feasible solution. We will also assume that

$$A = \left[ \begin{array}{cccccc} e_1 & \dots & e_m & y_{m+1} & \dots & y_n \end{array} \right] \text{ and } b = y_{n+1}.$$

In this case we will have

$$x_{bfs} = \left[ \begin{array}{c} y_{n+1} = b \\ 0_{n-m} \end{array} \right].$$

The cost associated with this basic feasible solution is

$$z_{bfs} = c^T x_{bfs} = c_B^T y_{n+1}.$$

Suppose $x$ is another feasible solution. How does the cost $c^T x$ compare with

$c^T x_{bfs}$. As $x$ is feasible $Ax = b = y_{n+1}$. Thus

$$x_1 a_1 + \ldots x_m a_m + x_{m+1} y_{m+1} + \ldots + x_n y_n = y_{n+1}.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{m} x_i e_i &= y_{n+1} - \sum_{i=m+1}^{n} x_i y_i \\
\Rightarrow c_B^T\left(\sum_{i=1}^{m} x_i e_i\right) &= c_B^T(y_{n+1}) - c_B^T\left(\sum_{i=m+1}^{n} x_i y_i\right) \\
\Rightarrow \sum_{i=1}^{m} x_i c_i &= z_{bfs} - \sum_{i=m+1}^{n} x_i \overbrace{c_B^T y_i}^{=:z_i} \\
\Rightarrow \sum_{i=1}^{m} x_i c_i &= z_{bfs} - \sum_{i=m+1}^{n} x_i z_i \\
\Rightarrow \sum_{i=1}^{n} x_i c_i &= z_{bfs} - \sum_{i=m+1}^{n} x_i z_i + \sum_{i=m+1}^{n} x_i c_i \\
\Rightarrow \underbrace{\sum_{i=1}^{n} x_i c_i}_{c^T x} &= \underbrace{z_{bfs}}_{c^T x_{bfs}} + \underbrace{\sum_{i=m+1}^{n} x_i (c_i - z_i)}_{difference\ in\ cost} \\
\Rightarrow \underbrace{\sum_{i=1}^{n} x_i c_i}_{c^T x} - \underbrace{z_{bfs}}_{c^T x_{bfs}} &= \underbrace{\sum_{i=m+1}^{n} x_i (c_i - z_i)}_{difference\ in\ cost}
\end{aligned}$$

Thus if $c_i - z_i \geq 0$ for all $i = m + 1, \ldots, n$ then $x_{bfs}$ is optimal.

# The Simplex Method

Consider the following SLP.

$$
\begin{array}{llllllllll}
\text{Minimize} & c_1x_1 & + & c_2x_2 & + & \dots & + & c_nx_n & & \\
\text{subject to:} & & & & & & & & & \\
& a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
& a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\
& \vdots & & & + & & & & & \\
& a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \\
& x_1 & , & x_2 & , & \dots & , & x_n & \geq & 0
\end{array}
$$

Lets assume that the matrix $A$ has rank $m$. We will assume that the matrix $A$ is such that $a_i = e_i, \ i = 1, \dots, m$ and $a_i = y_i, \ i = m+1, \dots, n$. We will denote the vector $b$ by $y_{n+1}$. Thus we have

$$\begin{array}{ll}\min & \begin{bmatrix} c_1 & c_2 & \cdots & c_m & c_{m+1} & \cdots & c_n \end{bmatrix} x \\ \text{subject to} & \end{array}$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_{1n+1} \\ y_{2n+1} \\ \vdots \\ y_{mn+1} \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_m & x_{m+1} & \cdots & x_n \end{bmatrix} \geq 0$$

Lets assume that $y_{1n+1}, \ldots, y_{mn+1} \geq 0$. Then for the above table a basic feasible solution is given by

$$x_1 = y_{1n+1}, \ldots, x_m = y_{mn+1},\ x_{m+1} = 0, \ldots, x_n = 0.$$

Suppose it is decided that $x_q$ will enter the basic set.

Consider the table:

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} \\
0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} \\
\vdots & & & & & & \\
0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn}
\end{bmatrix}
\quad
\begin{bmatrix}
y_{1n+1} \\
y_{2n+1} \\
\vdots \\
y_{mn+1}
\end{bmatrix}
$$

$$
\begin{bmatrix} c_1 & c_2 & \cdots & c_m & c_{m+1} & \cdots & c_n \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}
$$

Suppose we do the following operation:

$$
[\text{ row } (m+1)] \leftarrow [\text{ row } (m+1)] - c_1 [\text{ row } 1] - c_2 [\text{ row } 2] - \ldots c_m [\text{ row } m].
$$

Then we have the table: Consider the table:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} \\ \vdots & & & & & & \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} \end{bmatrix} \qquad \begin{bmatrix} y_{1n+1} \\ y_{2n+1} \\ \vdots \\ y_{mn+1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \sum_{i=1}^{m} c_i y_{im+1} & \cdots & \sum_{i=1}^{m} c_i y_{in} \end{bmatrix} \quad \begin{bmatrix} -\sum_{i=1}^{m} c_i y_{in+1} \end{bmatrix}$$

Note that

$$r_i = c_i - c_B^T y_i \text{ for all } i = 1, \ldots, n \text{ and } r_{n+1} = -\sum_{i=1}^{m} c_i y_{in+1} = -z_{bfs}.$$

Note that if any other feasible solution has cost $c^T x$ then

$$z - z_{bfs} = \sum_{i=m+1}^{n} c_i r_i. \tag{5}$$

- Determining the variable that will enter:

  First determine

  $$r_q = \min\{r_i, i = 1, \ldots, n\}.$$

  If

  $$r_q \geq 0,$$

  then the current bfs is the optimal solution as the cost of any other feasible solution is greater than or equal to $z_{bfs}$ (see Equation 5).

  If $r_q < 0$ then $q$ is chosen as the variable to become basic.

- Determining the basic variable that will leave the basic set:

Note that the bfs is given by

$$x_{bfs} = \begin{bmatrix} y_{n+1} \\ \mathbf{0} \end{bmatrix}.$$

Also note that

$$y_j = y_{1j}e_1 + y_{2j}e2 + \ldots + y_{mj}e_m = y_{1j}y_1 + y_{2j}e_2 + \ldots + y_{mj}y_m.$$

Suppose $q$ enters the basic set. Suppose we denote the new solution to be $\bar{x}$. Then as $A\bar{x} = b$ it follows that

$$\bar{x}_1 e_1 + \bar{x}_2 e_2 + \ldots + \bar{x}_m e_m + \epsilon y_q = y_{n+1}.$$

Thus

$$\sum_{i=1}^{m} (\bar{x}_i + \epsilon y_{iq} - x_i) e_i = 0.$$

Thus

$$\bar{x}_i = x_i - \epsilon y_{iq} \text{ for all } i = 1, \ldots, m \text{ and } \bar{x}_q = \epsilon.$$

That is

$$\bar{x} = \begin{bmatrix} y_{1n+1} - \epsilon y_{1q} \\ \vdots \\ y_{mn+1} - \epsilon y_{mq} \\ 0 \\ \vdots \\ 0 \\ \epsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix} = .$$

To be feasible $\bar{x} \geq 0$.

If $y_{iq} \leq 0$ for all $i = 1, \ldots, m$ then any $\epsilon > 0$ does not violate feasibility and the feasible set is unbounded. Note that in this case as $r_q < 0$ and the cost of making the non-basic $q$ variable to take a nonzero value $\epsilon$ is from Equation (5) is

$$z - z_{bfs} = r_q \epsilon.$$

As $\epsilon \geq 0$ can be arbitrarily large without violating feasibility we conclude that the SLP has no solution and the minimum value is $-\infty$. Thus if

$$y_{iq} \leq 0 \text{ for all } i = 1, \ldots, m$$

then one can stop the Simplex algorithm and conclude that the optimal value is $-\infty$.

If $y_{iq} > 0$ for some $i_0 \in \{1, \ldots, m\}$ then let

$$p = arg[\min_{i=1,\ldots,m} \{\frac{y_{in+1}}{y_{iq}} | y_{iq} > 0\}] \text{ and } \epsilon = \min_{i=1,\ldots,m} \{\frac{y_{in+1}}{y_{iq}} | y_{iq} > 0\}.$$

$p$ is the basic variable that will leave the basic set to be replaced by $q$ as the basic variable.

If the initial bfs is not degenerate then $y_{in+1} > 0$ for all $i = 1, \dots m$. Thus, $\epsilon > 0$ and from Equation (5) as $r_q < 0$ and $\epsilon > 0$ the new cost $z < z_{bfs}$. Thus if all bfs are non-degenerate then at every time the simplex table is updated the cost strictly decreases and thus the same solution cannot be visited twice. Thus there is no cycling in the iterations.

Update the table by the following operations

$$\begin{array}{lcll} y_{ij} & \leftarrow & y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} & \text{if } i \neq p \\ y_{pj} & \leftarrow & \frac{y_{pj}}{y_{pq}} & \end{array}.$$

In other words

$$\begin{array}{lcll} [\text{row } \ i] & \leftarrow & [\text{row } \ i] - \frac{y_{iq}}{y_{pq}} [\text{row } \ p] & \text{if } i \neq p \\ \text{row } \ p] & \leftarrow & \frac{1}{y_{pq}} [\text{row } \ p] & \end{array}.$$

Note that with this operation

$$y_i = e_i \text{ for all } i \in [\{1, 2, \ldots, p-1, p \ldots, m\} \cup \{q\}].$$

# The Simplex Algorithm

- (Step 1) Find index $q$ such that

$$r_q = \min\{r_j | j = 1, \ldots, n.\}$$

If $r_q \geq 0$ STOP. The current basic feasible solution is the optimal solution.

- (Step 2) Let $r_q$ be the solution in Step 1 with $r_q < 0$.

1. If $y_{iq} \leq 0$ for all $i = 1, \ldots, m$ then STOP. There is no optimal solution and the optimal value is $-\infty$.
2. If there exists $i_0$ such that $y_{i_0 q} > 0$ then let

$$\epsilon = \min\{\frac{y_{in+1}}{y_{iq}} | y_{iq} > 0\}$$

and let

$$p = arg[\min\{\frac{y_{in+1}}{y_{iq}}|y_{iq} > 0\}].$$

- (Step 3) Update the table by the following operations

$$\begin{array}{lcll} y_{ij} & \leftarrow & y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} & \text{if } i \neq p \\ y_{pj} & \leftarrow & \frac{y_{pj}}{y_{pq}} & \end{array}.$$

  A new basic feasible solution is obtained

- (Step 4) Update the relative cost vector to assure that all the relative cost with respect to basic variables are zero.

  Return to Step 1

**Theorem 7.** *The simplex algorithm will yield an optimal basic feasible solution in a finite number of steps if the SLP has any optimal solution and all basic feasible solutions are non-degenerate.*

**Proof:** Follows from the fact that there are finite number of basic feasible solutions and that the simplex algorithm at each iteration yields a new basic feasible solution that has a cost strictly smaller than the previous iteration cost (if all basic feasible solutions are non-degenerate there is no cycling).

∎

# Revised Simplex: Matrix Method

Let the SLP be given by

$$\mu = \min\{c^T x | Ax = b,\ x \geq 0\}$$

where $A \in R^{m\times n}$ and $b \in R^m$.

Suppose the first $m$ columns of $A$ are independent. Let

$$B := \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \text{ and } D := \begin{bmatrix} a_{m+1} & a_{m+2} & \cdots & a_n \end{bmatrix}.$$

With this definition we have

$$A = \begin{bmatrix} B & D \end{bmatrix}.$$

Partition any feasible $x$ according to

$$x = \begin{bmatrix} x_B \\ x_D \end{bmatrix}.$$

Let the basic feasible solution associated with $B$ be given by

$$\bar{x} = \begin{bmatrix} \bar{x}_B \\ \bar{x}_D \end{bmatrix}$$

where

$$\bar{x}_B = B^{-1}b \text{ and } \bar{x}_D = 0.$$

If

$$x = \begin{bmatrix} x_B & x_D \end{bmatrix}$$

is feasible then from $Ax = b$ it follows that

$$\begin{bmatrix} B & D \end{bmatrix} \begin{bmatrix} x_B & x_D \end{bmatrix} = b.$$

Thus

$$Bx_B + Dx_D = b.$$

Thus

$$x_B = B^{-1}b - B^{-1}Dx_D = \bar{x}_B - B^{-1}Dx_D.$$

The cost associated with this feasible solution is

$$\begin{aligned}
z = c^T x &= \begin{bmatrix} c_B^T & c_D^T \end{bmatrix} \begin{bmatrix} B & D \end{bmatrix} \\
&= c_B^T x_B + c_D^T x_D \\
&= c_B^T (B^{-1}b - B^{-1}Dx_D) + c_D^T x_D \\
&= c_B^T \bar{x}_B + (c_D^T - c_B^T B^{-1} D) x_D \\
z &= z_{bfs} + (c_D^T - c_B^T B^{-1} D) x_D \\
z - z_{bfs} &= (c_D^T - c_B^T B^{-1} D) x_D \\
&= r_D^T x_D
\end{aligned}$$

where $r_D^T = (c_D^T - c_B^T B^{-1} D)$.

Thus the following algorithm can be followed

- (Step 1): Compute
$$r_D^T = (c_D^T - c_B^T B^{-1} D).$$
If $r_D \geq 0$ STOP. The current solution is optimal.

- (Step 2) Let $q$ be the most negative element of $r_D$. $a_q$ will enter the basic set.

- (Step 3) Let
$$y_q = B^{-1} a_q$$
the coordinate vector of $a_q$ in the basis given by $B$.

- (Step 4) If $y_{iq} \leq 0$ for all $i = 1, \ldots, m$ STOP. The SLP has no solution and the optimal value is $-\infty$. Else calculate

$$p = arg[\min\{\frac{y_{in+1}}{y_{iq}} | y_{iq} > 0\}]$$

where

$$y_{n+1} = B^{-1}b.$$

Replace the vector $a_p$ in $B$ by $a_q$.

Go to step 1.