Homework 2

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1. Let $\beta$ be a $n$ dimensional random vector of zero mean and positive-definite covariance matrix $Q$. Suppose measurements of the form $y = W\beta$ are made where the rank of $W$ is $m$. If $\hat{\beta}$ is the linear minimum variance estimate of $\beta$ based on $y$, show that the covariance of the error $\beta - \hat{\beta}$ has rank $n - m$.

2. Assume that the measurement vector $y$ is obtained from $\beta$ by $y = W\beta + \epsilon$ where $E(\beta\beta') = R$, $E(\epsilon\epsilon') = Q$, $E(\beta\epsilon') = S$. Show that the minimum variance estimate of $\beta$ based on $y$ is

$$\hat{\beta} = (RW + S)(WRW' + WS + S'W' + Q)^{-1}y.$$ 

3. Let $\beta$ and $y$ be random vectors with $E(\beta) = \beta_0$ and $E(y) = y_0$ with finite covariance matrices. Show that the minimum variance estimate of $\beta$ of the form

$$\hat{\beta} = Ky + b$$

where $b$ is a constant vector is

$$\hat{\beta} = \beta_0 + E[(\beta - \beta_0)(y - y_0)']\{E[(y - y_0)(y - y_0)']\}^{-1}(y - y_0).$$

4. Let $\hat{\beta} = Ky$ be the minimum variance estimate of a random vector $\beta$ based on the random vector $y$. Show that

$$E[(\beta - \hat{\beta})(\beta - \hat{\beta})'] = E[\beta\beta'] - E[\hat{\beta}\hat{\beta}'].$$

5. **Stochastic Sampling Theorem** A continuous time zero-mean scalar stationary process $\{y(t), -\infty < t < \infty\}$ is called band limited if its power spectral density function defined as the Fourier Transform of its autocorrelation function:

$$S_y(j\omega) := F\{R_y(\tau)\} = F\{< y(t), y(t - \tau) >\}$$

is band limited, that is for some $W$, $S_y(j\omega) = 0$ for all $|\omega| > W$. The well known deterministic sampling theorem shows that $R_y(t)$ can be recovered from its samples through the formula

$$R_y(t - a) = \sum_{n=-\infty}^{\infty} \text{sinc}[W(t - nT)]R_y(nT - a), \text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

for any $a$ provided the sampling rate is such that $T < \pi/W$. 

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• Show that the minimum variance linear estimator of \( y(t) \) where \( t \) is a fixed number given the samples \( \{y(nT)\} \) can be written as
\[
\hat{y}(t) = \sum_{n=-\infty}^{\infty} \text{sinc}(W(t-nT))y(nT).
\]

• Show that the associated error is zero. Interpret the result.

6. Uncertain models
Consider a noisy measurement
\[
y = (h + \delta h)x + v
\]
where the scalar real-valued quantities \( \{x, v, \delta h\} \) are zero-mean independent random variables with the variances \( \{\sigma^2_x, \sigma^2_v, \sigma^2_{\delta h}\} \). The variable \( h \) is a known real scalar coefficient and \( \delta h \) represents the uncertainty in \( h \). Define the signal-to-noise ratio \( \text{SNR} = \frac{\sigma^2_x}{\sigma^2_v} \). Determine the mvle of \( x \) based on \( y \) and the resulting error. Express your answer in terms of \( \{h, \sigma^2_{\delta h}, \text{SNR}, y\} \).

7. Defective measurement sensors:
Consider a zero-mean random variable \( x \) with variance \( \Pi_0 \) and two possible measurements of \( x \) say
\[
y_1 = H_1x + v_1, \quad y_2 = H_2x + v_2,
\]
where \( \{v_1, v_2\} \) are zero mean uncorrelated sensor noise with variances \( R_1 \) and \( R_2 \) respectively. They are uncorrelated with \( x \). One of the measurement sensors is defective and it is either sensor 1 with probability \( 1 - p \) and sensor 2 with probability \( p \). The measurement that is used is therefore either \( y_1 \) with probability \( p \) or \( y_2 \) with probability \( (1-p) \). Denote this measurement by \( z \). Find the mvle estimator of \( x \) given \( z \) and the corresponding error. Comment on the special case \( H_1 = H_2 =: H \).

8. Separation of signal and structured noise
Consider the model
\[
y = Hx + S\theta + v
\]
where \( v \) is a zero-mean additive noise with unit variance and \( \{x, \theta\} \) are unknown constant vectors. \( H \) and \( S \) are known \( N \times n \) and \( N \times m \) matrices such that \( [H \ S] \) has full rank and \( N \geq m + n \). The term \( S\theta \) can be interpreted as structured uncertainty that is known to lie in the column span of \( S \) and \( s = Hx \) denotes the desired signal corrupted by \( S\theta \) and \( v \). We wish to estimate \( Hx \) from \( y \) and hence separate \( Hx \) from \( S\theta \).

• Define the column vector \( z = (x \ \theta)' \). Find \( \hat{z} \) the mvle of \( z \) based on \( y \).
• Partition \( \hat{z} = \{\hat{x} \ \hat{\theta}\} \). Determine \( \hat{s} = H\hat{x} \), the estimate of \( s \). Show that \( \hat{s} = \mathcal{E}_{HS}y \) where
\[
\mathcal{E}_{HS} = \mathcal{P}_H[I - S(S^*\mathcal{P}_{H}^\perp S)^{-1}S^*\mathcal{P}_{H}^\perp] = H(H^*\mathcal{P}_{S}^\perp H)^{-1}H^*\mathcal{P}_{S}^\perp
\]
with \( \mathcal{P}_{H}^\perp = I - \mathcal{P}_H \) and \( \mathcal{P}_{S}^\perp = I - \mathcal{P}_S \) where \( \mathcal{P}_H \) and \( \mathcal{P}_S \) denote the orthogonal projections onto the column span of \( H \) and \( S \) respectively.
• Conclude that \( \mathcal{E}_{HS}S = 0 \) and provide a geometric meaning for \( \mathcal{E}_{HS} \).
Let \( \tilde{s} = s - \hat{s} \). Show that \( E[\tilde{s}\tilde{s}^*] = \mathcal{E}_H S E_{HS}^* \).

9. (Multiplicative Noise) Let \( y = (1 + v)x \) where \( x \) and \( v \) are zero mean independent random variables. Only the variance of the noise \( v \) is known say \( \sigma_v^2 \). Determine the mve of \( x \) given \( y \). Show the error variance is smaller than the variance of \( x \).

10. (Decision Feedback Equalizers) Consider Figure 1 where a sequence \( x(i) \) is being transmitted over a channel with FIR given by \( C(z) = c_0 + c_1 z^{-1} + \ldots + c_M z^{-M} \). The channel output is corrupted by additive zero mean noise \( v(i) \) that is uncorrelated with the input sequence \( \{x(j)\} \). The main objective is to reconstruct the symbol sequence \( x(i) \).

![Diagram](https://via.placeholder.com/150)

Figure 1:

For this purpose the signal \( y(.) \) is fed to a Feedforward filter \( F(z) = f_0 + f_1 z^{-1} + \ldots + f_N z^{-N} \). The signal \( z(.) \) is used to make the decision which is given by \( \hat{x}(i) \). It is assumed that the architecture works and therefore \( x(i) = \hat{x}(i) \). The feedback filter \( B(z) \) is given by \( b_1 z^{-1} + \ldots + b_N z^{-Q} \).

Let

\[
y := \{y(i), \ldots, y(i - N)\}^T, \quad \mathbf{x} = \{x(i), \ldots, x(i - M - N)\}^T
\]

and

\[
v := \{v(i), \ldots, v(i - N)\}^T, \quad x := \{x(i), \ldots, x(i - Q)\}^T.
\]

Define the row vectors

\[
f = [f_0 \ f_1 \ \ldots \ f_N] \quad b = [1 \ b_1 \ b_2 \ \ldots \ b_q].
\]

- Show that \( y = H \mathbf{x} + v \) for some \( H \). Determine \( H \) and find the covariance matrix \( R_y \) of \( y \) in terms of \( H, R_x, R_v \).
- Show that \( \hat{x}(i) = bx - fy \) where \( \hat{x} := x(i) - z(i) \).
- Show that the optimal choice \( \{f_{opt}, b_{opt}\} \) that minimize the error variance \( ||\hat{x}||^2 \) are given by

\[
f_{opt} = b_{opt} R_{xy} R_y^{-1}, \quad b_{opt} = \frac{\text{first row of } R_{\Delta}^{-1}}{[R_{\Delta}^{-1}]_{00}}
\]

where \( R_\Delta = R_x - R_{xy} R_y^{-1} R_{yx} \) and \( [R_{\Delta}^{-1}]_{00} \) denotes the first diagonal entry of \( R_\Delta^{-1} \).
11. Let \( y(i) = x + v(i) \) where \( x \) and \( v(i) \) are independent random variables with \( v(i) \) being a white-noise Gaussian process with zero mean and unit variance and \( x \) takes the values 1 and \(-1\) with equal probability.

- Show that the minimum variance estimate (not necessarily linear) of \( x \) given \( N \) observations \( \{y(0), \ldots, y(N-1)\} \) is
\[
\hat{x}_N = \tanh(\sum_{i=0}^{N-1} y(i)).
\]

- Assume \( x \) takes value 1 with probability \( p \) and value \(-1\) with probability \( 1 - p \). Show that the minimum variance estimate (not necessarily linear) of \( x \) given \( N \) observations \( \{y(0), \ldots, y(N-1)\} \) is
\[
\hat{x}_N = \tanh\left(\frac{1}{2} \ln\left(\frac{p}{q} + \sum_{i=0}^{N-1} y(i)\right)\right).
\]

12. Suppose \( y = x + v \) where \( x \) and \( v \) are independent real-valued random variables with exponential distribution with parameters \( \lambda_1 \) and \( \lambda_2 \), \( \lambda_1 \neq \lambda_2 \). Show that the minimum variance estimate of \( x \) given \( y \) is
\[
\hat{x} = \frac{1}{\lambda_1 - \lambda_2} - \frac{e^{-\lambda_1 y}}{e^{-\lambda_2 y} - e^{-\lambda_1 y}}.
\]

(The exponential pdf with parameter \( \lambda \) is given by \( f(x) = \lambda e^{\lambda x} \) for \( x \geq 0 \).)