EE8950
Homework # 5

1. Apply the KKT theorem to show that the optimal solution to the following problem if it exists is FIR

\[ \mu = \inf \{ c_1 \| \phi \|_1 + c_2 \| \phi_2 \|_2 : A\phi = b, \ \phi \in \ell_1 \} \]

where \( \ell_1 \) is the Banach space of all absolutely summable sequences and \( A : \ell_1 \to \mathbb{R}^n \).

2. Consider \( X = C[0,1] \) the space of continuous functions on the interval \([0,1]\). Let \( f : X \to \mathbb{R} \) be defined by

\[ f(x) = \max_{0 \leq t \leq 1} x(t) \]

for any \( x \in X \).

(a) Show that the Gateaux derivative of \( f \) exists at \( x \in C[0,1] \) if \( x \) has a unique maximum at \( t_0 \) in \([0,1]\). Find the Gateaux derivative.

(b) Show that the Gateaux derivative does not exist if \( x \) achieves a maximum at two different points \( t_0 \) and \( t_1 \) in \([0,1]\).

3. Consider \( X = C[0,1] \) the space of continuous functions on the interval \([0,1]\). Let \( f : X \to \mathbb{R} \) be defined by

\[ f(x) = \int_0^1 [ax^2(t) + bx(t)] dt \]

for any \( x \in X \).

(a) Show that the Gateaux derivative of \( f \) exists at \( x \in C[0,1] \). Find the Gateaux derivative.

(b) Show that the Gateaux derivative is continuous and therefore Frechet differentiable. What is the Frechet derivative.

4. Consider \( X = C[0,1] \) the space of continuous functions on the interval \([0,1]\). Let \( f : X \to \mathbb{R} \) be defined by

\[ f(x) = \int_0^1 |x(t)| dt. \]
(a) Show that the Gateaux derivative of $f$ exists at $x \in C[0,1]$ if $x$ is such that $|x(t)| \neq 0$ for all $t \in [0,1]$. Find the Gateaux derivative.

(b) Comment on when the Gateaux derivative will not exist.

Remark: Assume that the limits and integration operators are interchangeable; this need not be assumed as it can be proven.

5. Consider $X = C[0,1]$ the space of continuous functions on the interval $[0,1]$. Let $f : X \to \mathbb{R}$ be defined by

$$f(x) = [x(1/2)]^2.$$ 

(a) Find the Gateaux derivative of $f$ at any $x \in C[0,1]$.

(b) Is $f$ Frechet differentiable