Waterbed Effect

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8:15 AM

Theorem: Let L have a relative degree greater than or equal to 2 and let L have N_p poles in the rhp given by $p_1, p_2, \dots p_{N_p}$. If the closed-loop system is stable then.

$$S = \frac{1}{1+L} \quad \text{has to satisfy}$$

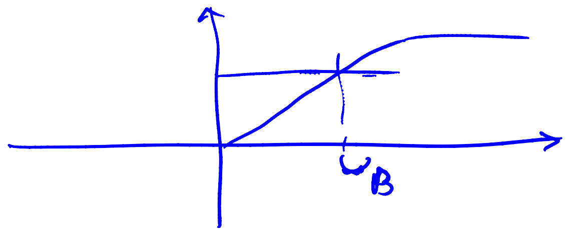
$$\int_0^\infty \ln |S(j\omega)| \, d\omega = \underbrace{\pi \sum_{i=1}^{N_p} Re(p_i)}_{\geq 0}.$$

Remark: Suppose the stabilizing controller K is such that $|S(j\omega)| \leq \varepsilon_p$ for $\omega \in [0, \omega_B]$

$$\pi \sum_{i=1}^{N_p} Re(p_i) = \int_0^\infty \ln |S(j\omega)| \, d\omega = \int_0^{\omega_B} \ln |S(j\omega)| \, d\omega + \int_{\omega_B}^\infty \ln |S(j\omega)| \, d\omega$$

$$\Rightarrow \quad \pi \sum_{i=1}^{N_p} Re(p_i) \leq \int_0^{\omega_B} \ln \varepsilon_p \, d\omega + \int_{\omega_B}^\infty \ln |S(j\omega)| \, d\omega$$

$$\Rightarrow \quad \int_{\omega_B}^\infty \ln |S(j\omega)| \, d\omega \geq \pi \sum_{i=1}^{N_p} Re(p_i) + (\ln \varepsilon_p)\, \omega_B$$



Proof: $S = \frac{1}{1+L} = \frac{1}{1+\frac{n_L}{d_L}} = \frac{d_L}{d_L + n_L}$

$\therefore$ the rhp poles of L are the rhp zeros of S.

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$S(z) = 1.$

$$S_{ap}(s) = \prod_{i=1}^{N_p} \frac{s - p_i}{s + \bar{p}_i}$$

$$\boxed{G = G_{ap}\, G_{mp} \qquad \prod_{i=1}^{\hat{}} \frac{s - z_i}{s + \bar{z}_i}}$$

From an ~~eta~~ earlier theorem; $s_0 = x + jy$; $x > 0$

$$\ln|S_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{x}{x^2 + (\omega - y)^2}\, d\omega$$

Let $y = 0$; $x > 0$

$$\Rightarrow x \ln|S_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{x^2}{x^2 + \omega^2}\, d\omega$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \ln|S(j\omega)| \frac{x^2}{x^2 + \omega^2}\, d\omega$$

$$\Rightarrow \lim_{x \to \infty} x \ln|S_{mp}(s_0)| = \lim_{x \to \infty} \frac{2}{\pi}\int_{0}^{\infty} \ln|S(j\omega)| \frac{x^2}{x^2 + \omega^2}\, d\omega$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \ln|S(j\omega)|\, d\omega.$$

$$\therefore \int_0^{\infty} \ln|S(j\omega)|\, d\omega = \frac{\pi}{2} \lim_{x \to \infty} x \ln|S_{mp}(x)|$$

$$\lim_{x \to \infty} x \ln|S_{mp}(x)| = \lim_{x \to \infty} x \ln\left|\frac{S(x)}{S_{ap}(x)}\right|$$

$$= \lim_{x \to \infty} x \ln|S(x)| + \lim_{x \to \infty} x \ln|S_{ap}^{-1}(x)|$$

$$\lim_{x \to \infty} x \ln|S(x)| = \lim_{x \to \infty} x \ln\left|\frac{1}{1 + L(x)}\right| \qquad ; L = \frac{n_L}{d_L}$$

$$= \lim_{x \to \infty} x \ln\left|\frac{1}{1 + \frac{1}{x^k}}\right| \qquad \text{for } x \gg 1,\ |L| \cong \left|\frac{1}{x^k}\right|$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha} \ln\left|\frac{1}{1 + \alpha^k}\right|$$

$$= \lim_{\alpha \to 0} \frac{d}{d\alpha} \ln\left|\frac{1}{1 + \alpha^k}\right| \rightarrow \text{L'Hospital rule}$$

$$= \lim (1 + \alpha^k)^{-1} \cdot k\, \alpha^{k-1} \qquad ; k \geq 2$$

$$\therefore \lim_{\substack{\alpha \to 0 \\ x \to \infty}} x \ln |S(x)| = 0$$

$$\lim_{x \to 0} x \ln \left| S_{ap}^{-1}(x) \right| = \lim_{x \to 0} x \ln \left| \prod_{i=1}^{N_p} \frac{x + \bar{p}_i}{x - p_i} \right|$$

$$= \lim_{x \to \infty} x \sum_{i=1}^{N_p} \ln \left| \frac{x + \bar{p}_i}{x - p_i} \right|$$

$$= \sum_{i=1}^{N_p} \left[\lim_{x \to \infty} x \ln \frac{|x + \bar{p}_i|}{|x - p_i|} \right]$$

let $p_i = \gamma_i + j\beta_i \quad ; \quad \gamma_i \geq 0$

$$\lim_{x \to 0} x \ln \left| \frac{x + \bar{p}_i}{x - p_i} \right|$$

$$= \lim_{x \to 0} x \ln \left| \frac{x + \gamma_i - j\beta_i}{x - \gamma_i - j\beta_i} \right|$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha} \ln \frac{|1 + \alpha(\gamma_i - j\beta_i)|}{|1 - \alpha(\gamma_i + j\beta_i|}$$

$$= \lim_{\alpha \to 0} \frac{d}{d\alpha} \ln \left| \frac{1 + \alpha(\gamma_i - j\beta_i)}{1 - \alpha(\gamma_i + j\beta_i)} \right|$$

$$= \lim_{\alpha \to 0} \frac{d}{d\alpha} \ln \sqrt{\frac{(1 + \alpha\gamma_i)^2 + \alpha^2\beta_i^2}{(1 - \alpha\gamma_i)^2 + \alpha^2\beta_i^2}}$$

$$= \lim_{\alpha \to 0} \sqrt{\frac{(1 - \alpha\gamma_i)^2 + \alpha^2\beta_i^2}{(1 + \alpha\gamma_i)^2 + \alpha^2\beta_i^2}} \left\{ \frac{1}{\sqrt{(1 - \alpha\gamma_i)^2 + \alpha^2\beta_i^2}} \frac{1}{2}\left((1 + \alpha\gamma_i)^2 + \alpha^2\beta_i^2\right)^{-1/2} \left((2(1 + \alpha\gamma_i)\gamma_i + 2\alpha\beta_i^2\right) \right.$$

$$\left. \left(\sqrt{(1 + \alpha\gamma_i)^2 + \alpha^2\beta_i^2}\right) - \frac{1}{2}\left((1 - \alpha\gamma_i)^2 + \alpha^2\beta_i^2\right)^{-3/2} \left[2(1 - \alpha\gamma_i)(-\gamma_i) + 2\alpha\beta_i^2\right] \right\}$$

$$= 1 \left\{ 1 \cdot \frac{1}{2} \cdot 1\, 2\gamma_i + \frac{1}{2} 2\gamma_i \right]$$

$$= 2\gamma_i$$

$$\therefore \int_0^\infty \ln |S(j\omega)| \, d\omega = \frac{\pi}{2} \lim_{x \to 0} x \ln |S_{mp}(x)|$$

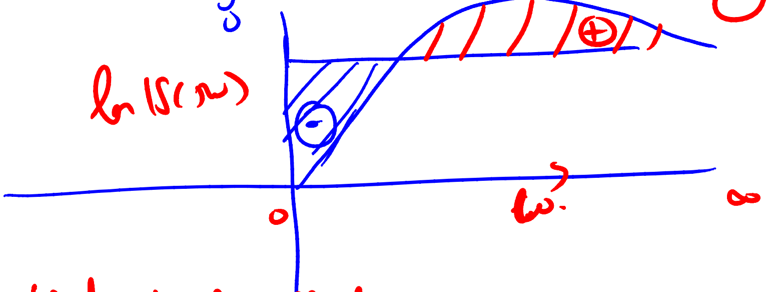
$$= 0 + \frac{\pi}{2} \sum_{i=1}^{N_p} (2\gamma_i)$$

$$= \pi \sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$

— x — $\quad$ QED

In the case of a stable L.

$$\int_0^\infty \ln|S(j\omega)|\, d\omega = 0 \qquad \ominus = \oplus$$



First Waterbed effect.

— x —

Waterbed Effect II:

Theorem: Let L have N_p poles in the rhp given by $p_1, p_2, \ldots, p_{N_p}$. Let $z = x + jy$ be any zero of L in the strict rhp. If the unity negative feedback system is stable then with $S = \frac{1}{1+L}$: the following has to be satisfied

$$\int_0^\infty \ln|S(j\omega)| \underbrace{\left[\frac{x}{x^2+(\omega-y)^2} + \frac{x}{x^2+(\omega+y)^2}\right]}_{\text{frequency dependent weight}} d\omega = \pi \sum_{i=1}^{N_p} \ln\left|\frac{z+\bar{p}_i}{z-p_i}\right|$$

Proof: Let $z = x + jy$

$$\ln|S_{mp}(z)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{x}{x^2+(\omega-y)^2}\, d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \ln|S(j\omega)| \frac{x}{x^2+(\omega-y)^2}\, d\omega$$

$$+ \frac{1}{\pi}\int_{-\infty}^{0} \ln|S(j\omega)| \frac{x}{x^2+(\omega-y)^2}\, d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \ln|S(j\omega)| \left[\frac{x}{x^2+(\omega-y)^2} + \frac{x}{x^2+(\omega+y)^2}\right] d\omega$$

$$\ln|S_{mp}(z)| = \ln\left|\frac{S(z)}{S_{ap}(z)}\right| = \ln|S(z)| + \ln|S_{ap}^{-1}(z)|$$

$$= \ln \left| \prod_{i=1}^{N_p} \frac{z+\bar{p}_i}{z-p_i} \right|$$

$$= \sum_{i=1}^{N_p} \ln \left| \frac{z+\bar{p}_i}{z-p_i} \right|$$

$$\int_0^\infty \ln|S(j\omega)| \underbrace{\left[\frac{x}{x^2+(\omega-y)^2} + \frac{x}{x^2+(\omega+y)^2}\right]}_{X(j\omega)} d\omega = \pi \sum_{i=1}^{N_p} \ln \left| \frac{z+\bar{p}_i}{z-p_i} \right|$$

Theorem: Suppose L has rhp poles and zeros at $p_1, p_2 \dots p_{N_p}$ and $z_1, z_2, \dots z_{N_z}$ respectively. If the closed-loop system is stable then

1. $\|w_p S\|_{H_\infty} \geq \max_j \left\{ |w_p(z_j)| \prod_{i=1}^{N_p} \left| \frac{z+\bar{p}_i}{z-p_i} \right| \right\}$

2. $\|w_T T\|_{H_\infty} \geq \max_i \left\{ |w_T(p_i)| \prod_{j=1}^{N_z} \left| \frac{\bar{z}_j+p_i}{z_j-p_i} \right| \right\}$

Proof: $\|W_p S\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} |W_p(j\omega)| \, |S(j\omega)|$

$$= \sup_{\omega \in \mathbb{R}} |W_p(j\omega) \, S_{ap}(j\omega) \, (S_{mp}(j\omega))|$$

$$= \sup_{\omega \in \mathbb{R}} |W_p(j\omega) S_{mp}(j\omega)|$$

$$= \sup_{Re(s) \geq 0} |W_p(s) \, S_{mp}(s)|$$

$$\geq |W_p(z_j) \, S_{mp}(z_j)| \quad \text{for any } z_j \text{ a zero of } L.$$

$$= |W_p(z_j) \, S(z_j) \, S_{ap}^{-1}(z_j)|$$

$$= \left| W_p(z_j) \prod_{i=1}^{N_p} \frac{z_j+\bar{p}_i}{z_j-p_i} \right|$$

$$\therefore \|W_p S\|_{H_\infty} \geq \left| W_p(z_j) \prod_{i=1}^{N_p} \frac{z_j+\bar{p}_i}{z_j-p_i} \right|$$

for any z_J a rhp zero of L

$$\therefore \; \|W_pS\|_{H_\infty} \geq \max_J \left| W_p(z_J) \prod_{i=1}^{N_p} \frac{z_J + \bar{p}_i}{z_J - p_i} \right|.$$

Remark: One conclusion is that $\|W_pS\|_{H_\infty}$ cannot have a low value

if $|z_J - p_i| << 1$ for any pair z_J, and p_i

$\omega_B < z/2$; $\omega_T > 2p$

- we need $|L| >> 1$. for $\omega \in [0, \omega_B]$

$|L| << 1$ for $\omega \in [\omega_T, \infty)$

$\omega_B < \omega_C < \omega_T$

$z/2 > 2p$

$z > 4p$