Fundamental Limitations.

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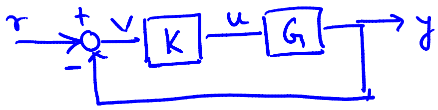


S: Sensitivity transfer function $= \frac{1}{1+GK}$

T: Complimentary transfer function $= \frac{GK}{1+GK}$

First fundamental limitation:

$$S + T = 1$$

$$S(j\omega) + T(j\omega) = 1 \qquad \forall\, \omega \in \mathbb{R}$$

Preliminaries on Complex Analysis:

Definitions: Analytic functions, holomorphic functions

A function $f: D \to \mathbb{C}$ is said to be analytic at $s_0 \in \mathbb{C}$ if $\left.\frac{df}{ds}\right|_{s=s_0}$ exists.

It is said to be holomorphic if it is analytic in the entire domain D.

Definition: (Rectifiable Curve, Simple Curve, closed Curve)

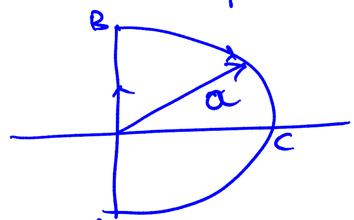
Γ is a rectifiable curve is there exists an interval $[a,b] \subset \mathbb{R}$ and function $\gamma: [a,b] \to \mathbb{C}$ such that $\Gamma = \{\gamma(x) : x \in [a,b]\}$

Γ is a Simple Curve if it does not intersect itself i.e.

$$\gamma(x) = \gamma(y) \Rightarrow x = y$$

Γ is a closed Curve if $\gamma(a) = \gamma(b)$ i.e. the "starting point" and the end point are the same.

Example:



Γ_{AB}

$\gamma_{ACB}: [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{C}$ defined by

$\gamma_{ACB}(\theta) = a e^{j\theta}$ defines the segment from A to B.

$$\gamma_{AB} : [-a, a] \longrightarrow \mathbb{C}$$

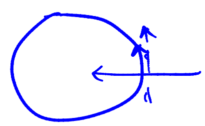
$$\gamma_{AB}(\omega) = J\omega.$$

Definition: Contour

A Contour is a union of rectifiable curves Γ_J with the end point of Γ_J being the starting point of Γ_{J+1}

Closed Contours and Simple contours are defined in an analogous manner

Definition: Positively oriented Contour :



Definition: (Integral) A function f that is continuous on a domain S the integral along a rectifiable curve $\Gamma \subset S$ is defined as

$$\int_\Gamma f(s)\, ds = \int_a^b f(\gamma(x)) \frac{d\gamma}{dx}\, dx.$$

where $\gamma : [a,b) \to \mathbb{C}$ defines the rectifiable curve Γ.

Intuition is define $s = \gamma$

$$ds = \frac{d\gamma}{dx} dx$$

as s varies from the start point of Γ to the end point of Γ, x varies from a to b.

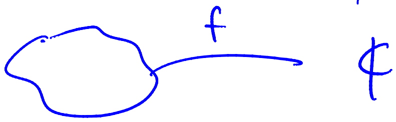
$$\int_\Gamma f(s)\, ds = \sum_{J=1}^{n} \int_{a_J}^{b_J} f(\gamma_J(x)) \frac{d\gamma_J(x)}{dx}\, dx.$$

Theorem: Maximum Modulus Principle.

If f is analytic in a domain Ω; and f is not a constant then $|f|$ does not attain its maximum value at an interior point of Ω.

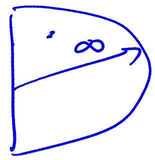
f

maximum value at an interior point of Ω.



A simple corollary of this result is that F is analytic inside the RHP

$$\|F\|_{H_\infty} := \sup_{\mathrm{Re}\, s \geq 0} |F(s)|.$$



$$= \sup_{\omega \in \mathbb{R}} |F(j\omega)|$$

Theorem: [Cauchy Theorem]:

Suppose f is analytic in a domain S that contains a closed contour Γ that is positively oriented

$$\int_\Gamma f(s)\, ds = 0$$

Furthermore if $s_0 \in S$

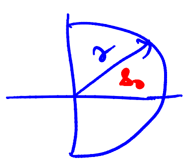
$$\frac{1}{2\pi j} \int_\Gamma \frac{f(s)}{(s - s_0)}\, ds = f(s_0)$$

Weighted Cauchy's Theorem:

Let F be analytic and of bounded magnitude on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$. Let $s_0 = x + jy$ for $x, y \in \mathbb{R}$ be such that $x > 0$. Then

$$F(s_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(j\omega) \frac{x}{x^2 + (\omega - y)^2}\, d\omega.$$

Proof: Lets consider the Nyquist Contour $D(r)$ with radius r that encloses s_0.



$$F(s_0) = \frac{1}{2\pi j} \int_{D(r)} \frac{F(s)}{s - s_0}\, ds$$

$-\bar{s}_0 = -(x - jy) = -x + jy \in$ Lhp and this point is not inside the Nyquist Contour

$$\frac{1}{2\pi j} \int_{D(r)} \frac{F(s)}{s + \bar{s}_0}\, ds = 0.$$

Lets consider

$$F(s_0) - 0 = \frac{1}{2\pi j} \int_{D(r)} \left[\frac{F(s)}{(s-s_0)} - \frac{F(s)}{s+\bar{s}_0} \right] ds \qquad s = j\omega$$

$$= -\frac{1}{2\pi j} \int_{-r}^{r} F(j\omega) \left[\frac{1}{j\omega - s_0} - \frac{1}{j\omega + \bar{s}_0} \right] j\,d\omega.$$

$I_1(r)$

$$+ \frac{1}{2\pi j} \int_{-\pi/2}^{\pi/2} F(re^{j\theta}) \left[\frac{1}{re^{j\theta} - s_0} - \frac{1}{re^{j\theta} + \bar{s}_0} \right] rje^{j\theta} d\theta$$

$I_2(r)$

$$I_2(r) = \frac{1}{2\pi j} \int_{-\pi/2}^{\pi/2} F(re^{j\theta}) \left[\frac{\cancel{re^{j\theta}} + \bar{s}_0 - \cancel{re^{j\theta}} + s_0}{(re^{j\theta} - s_0)(re^{j\theta} + \bar{s}_0)} \right] rje^{j\theta} d\theta$$

$$= \frac{1}{2\pi j} \int_{-\pi/2}^{\pi/2} F(re^{j\theta}) \left[\frac{2x}{(re^{j\theta} - s_0)(re^{j\theta} + \bar{s}_0)} \right] jre^{j\theta} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} F(re^{j\theta}) \frac{2x}{(e^{j\theta} - r^{-1}s_0)(e^{j\theta} + r^{-1}\bar{s}_0)} \frac{re^{j\theta}}{r^2} d\theta$$

$$|I_2(r)| = \left| \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} F(re^{j\theta}) \frac{1}{r} \frac{2x\, e^{j\theta}}{(e^{j\theta} - r^{-1}s_0)(e^{j\theta} + r^{-1}\bar{s}_0)} d\theta \right|$$

$$\therefore \le \frac{1}{2\pi r} \int_{-\pi/2}^{\pi/2} \left| F(re^{j\theta}) \right| \left| \frac{2x}{(e^{j\theta} - r^{-1}s_0)(e^{j\theta} + r^{-1}\bar{s}_0)} \right| d\theta$$

$$\le \frac{1}{2\pi r} \|F\|_{H_\infty} \underbrace{\int_{-\pi/2}^{\pi/2} \frac{|2x|}{|(e^{j\theta} - r^{-1}s_0)|\,|e^{j\theta} + r^{-1}\bar{s}_0|} d\theta.}_{\le C \text{ as for all } r \ge M}$$

$$\therefore \lim_{r\to\infty} |I_2(r)| \to 0$$

$$I_1(r) = -\frac{1}{2\pi j} \int_{-r}^{r} F(j\omega) \left(\frac{1}{j\omega - s_0} - \frac{1}{j\omega + \bar{s}_0} \right) j\,d\omega$$

$$= -\frac{1}{2\pi} \int_{-r}^{r} F(j\omega) \left[\frac{\cancel{j\omega} + \bar{s}_0 - \cancel{j\omega} + s_0}{(j\omega - s_0)(j\omega + \bar{s}_0)} \right] d\omega$$

$$= -\frac{1}{2\pi} \int_{-r}^{r} F(j\omega) \left[\frac{2x}{(j\omega - x - jy)(j\omega + x - jy)} \right] d\omega$$

$$= -\frac{1}{\pi} \int_{-r}^{r} F(j\omega) \frac{x}{(j(\omega - y))^2 - x^2} d\omega$$

$$= \frac{1}{\pi} \int_{-r}^{r} F(j\omega) \frac{x}{x^2 + (\omega - y)^2} d\omega.$$

as $r \to \infty$

$$\lim I_1(r) = \frac{1}{} \int^{\infty} F(j\omega)\ x\ d\omega.$$

$$\lim_{r\to\infty} \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \cdots \frac{x}{x^2+(\omega-y)^2}$$

$$F(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega)\,\frac{x}{x^2+(\omega-y)^2}\,d\omega.$$

QED

All-pass/Minimum Phase Factorization:

Every stable proper rational transfer function G admits a factorization of the form

$$G = G_{ap}\, G_{mp}$$

where G_{ap} and G_{mp} are all pass and minimum phase, stable transfer functions.

[$G_{ap}(s)$ is said to be all pass if $|G_{ap}(j\omega)| = 1$ $\forall \omega \in \mathbb{R}$

$G_{mp}(s)$ is said to minimum phase if it has no rhp zeros).

Proof:

$$G(s) = K\,\frac{(s-z_1)(s-z_2)(s-z_3)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_k)}$$

WLOG let $z_1, z_2 \cdots z_m \in \text{RHP}$

$$G(s) = \underbrace{\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s+\bar z_1)(s+\bar z_2)\cdots(s+\bar z_m)}}_{G_{ap}}\ \overbrace{\frac{K(s+\bar z_1)(s+\bar z_2)\cdots(s+\bar z_m)(s-z_{m+1})\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_k)}}^{G_{mp}}$$

$$G_{ap} = \prod_{i=1}^{m}\frac{s-z_i}{s+\bar z_i}\ ; \qquad \left|\frac{j\omega-(\alpha+j\beta)}{j\omega+(\alpha-j\beta)}\right| = 1$$

$$\therefore\ |G_{ap}(j\omega)| = 1$$

Lemma: Let $G(s)$ be a stable proper transfer function with a all-pass, minimum phase factorization $G = G_{ap}G_{mp}$. Let $s_0 = x+jy$ with $x>0$. Then

$$\log|G_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|G(j\omega)|\,\frac{x}{x^2+(\omega-y)^2}\,d\omega.$$

Proof: $F(s) := \log G_{mp}(s)$; This is an analytic function in the RHP

$$F(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega)\,\frac{x}{x^2+(\omega-y)^2}\,d\omega.$$

$$\underline{\operatorname{Re} F(s_0)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{Re}(F(j\omega))\,\frac{x}{x^2+(\omega-y)^2}\,d\omega$$

$$\underline{\underline{\mathrm{Re}\ F(s_0)}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{Re}(F(j\omega)) \frac{x}{x^2+(\omega-y)^2}\, d\omega$$

$$F = \lg G_{mp} \Rightarrow e^F = G_{mp}$$

$$\Rightarrow G_{mp} = e^{\mathrm{Re}(F)} e^{j\,\mathrm{Img}\,F}$$

$$\Rightarrow |G_{mp}| = e^{\mathrm{Re}(F)}$$

$$\Rightarrow \lg |G_{mp}| = \mathrm{Re}(F).$$

$$\lg|G_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \lg|G_{mp}(j\omega)| \frac{x}{(x^2+(\omega-y)^2)}\, d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \lg\left[|G_{mp}(j\omega)|\,|G_{ap}(j\omega)|\right] \frac{x}{x^2+(\omega-y)^2}\, d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \lg|G(j\omega)| \frac{x}{x^2+(\omega-y)^2}\, d\omega.$$

<u>Theorem</u>:

$$\underline{\underline{\int_0^{\infty} \ln|S(j\omega)|\, d\omega = \pi \sum_{i=1}^{N_p} \mathrm{Re}(P_i)}}$$

RHP Poles of L.