Coprime factorization for MIMO System

G_{22} and K which are MIMO Systems

→ $G_{22} = NM^{-1}$ where N and M stable ~~is~~ proper transfer Matrices is a rcf of G_{22} if ∃ stable proper transfer matrices $\tilde{X}$ and $\tilde{Y}$ such that

$$\tilde{X}M - \tilde{Y}N = I$$

→ $G_{22} = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorization if ∃ matrices X and Y such that

$$\tilde{M}X - \tilde{N}Y = I$$

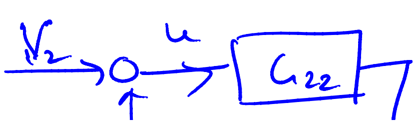
→ given that G_{22} is a transfer matrix (proper) then it always admits a doubly coprime factorization where ∃ matrices $M, N, \tilde{M}, \tilde{N}, \tilde{X}, \tilde{Y}$, ~~x~~ and Y all stable and proper such that

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

$\tilde{X}M - \tilde{Y}N = I$ ✓
$\tilde{X}Y - \tilde{Y}X = 0$ ✓
$-\tilde{N}M + \tilde{M}N = 0$ ✓
$-\tilde{N}Y + \tilde{M}X = I$ ✓

Says that M, N is a rcf
$Y\tilde{X}^{-1} = \tilde{X}^{-1}Y$
$NM^{-1} = \tilde{M}^{-1}\tilde{N}$
$\tilde{N}, \tilde{M}$ is a lcf of G_{22}.

with $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$.

→ 

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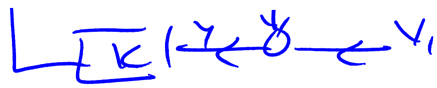


Internal Stability $\Leftrightarrow$ $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}$ is stable

→ Suppose $K = Y_1 X_1^{-1}$ is a $\hat{r}cf$ of K then the above interconnection is internally stable if and only

$$\begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix}^{-1} \in RH_\infty$$ [the Space of all stable proper transfer matrices].

→ A dcf of G_{22} yields a controller that will Stabilize the feedback interconnection of G_{22} and K

Pf:
$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$$

It follows that $\underbrace{\begin{bmatrix} M & Y \\ N & X \end{bmatrix}^{-1}} = \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \in RH_\infty$

$\therefore$ YX^{-1} is a stabilizing controller.

→ <u>Parametrization of all stabilizing controller</u>

→ Suppose $\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$ is a dcf of $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$. Then we see that

$$\begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \underbrace{\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}} \underbrace{\begin{bmatrix} M & Y \\ N & X \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}} = \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} \tilde{X} + Q\tilde{N} & -\tilde{Y} - Q\tilde{M} \end{bmatrix} \begin{bmatrix} M & Y + MQ \end{bmatrix} = I$$

$$\begin{bmatrix} -\tilde N & \tilde M \end{bmatrix} \begin{bmatrix} N & X+NQ \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} M & Y-MQ \\ N & X-NQ \end{bmatrix}^{-1} = \underbrace{\begin{bmatrix} \tilde X - Q\tilde N & -\tilde Y + Q\tilde M \\ -\tilde N & \tilde M \end{bmatrix}}_{\in RH_\infty.}$$

If $Q \in RH_\infty$

$\Rightarrow$ $\underline{(Y+MQ)(X+NQ)^{-1}}$ is a stabilizing controller.

→ from a dcf one can say that

$\{ (Y-MQ)(X-NQ)^{-1}$ is a Q stable $\}$

is a set of stabilizing controllers.

→ Suppose K is a stabilizing controller with a rcf $K = Y_1 X_1^{-1}$; then it follows that

$$\begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix}^{-1} \text{ is stable. where}$$

$\underline{\begin{bmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I}$ provides a dcf of G_{22}.

$$\begin{bmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{bmatrix} \begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix} = \begin{bmatrix} \tilde X M - \tilde Y N & \tilde X Y - \tilde Y X_1 \\ -\tilde N M + \tilde M N & -\tilde N Y_1 + \tilde M X_1 \end{bmatrix}$$

$$= \begin{bmatrix} I & (\tilde X Y - \tilde Y X_1) \\ 0 & D \end{bmatrix}$$

where $D := -\tilde N Y_1 + \tilde M X_1 = \underline{\tilde M X_1 - \tilde N Y_1}$

$Q := -(\tilde X Y_1 - \tilde Y X_1) D^{-1}$

$$\begin{bmatrix} \tilde X & -\tilde Y \\ \cdots \end{bmatrix} \begin{bmatrix} M & Y_1 \\ \cdots \end{bmatrix} = \begin{bmatrix} I & -QD \\ \cdots \end{bmatrix}$$

$-\tilde N \quad \tilde M \quad N \quad X_1 \quad 0 \quad D$

Note that $\underbrace{\begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix}^{-1}}_{\text{This inverse is stable}} = \underbrace{\begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix}^{-1}}_{H_\infty} \underbrace{\begin{bmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{bmatrix}^{-1}}_{H_\infty}$

$\therefore$ D^{-1} is a stable matrix. $(\tilde M X_1 - \tilde N Y_1)^{-1}$ is stable

$Q = -\underline{(\tilde X Y_1 - \tilde Y X_1)} D^{-1}$ is also

$$\begin{bmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{bmatrix}\begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix} = \begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix}$$

$$\therefore \underbrace{\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{bmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{bmatrix}}_{I}\begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix} = \underline{\begin{bmatrix} M & Y \\ N & X \end{bmatrix}}\begin{bmatrix} I & -QD \\ 0 & D \end{bmatrix}$$

$$\begin{bmatrix} M & Y_1 \\ N & X_1 \end{bmatrix} = \begin{bmatrix} M & (Y-MQ)D \\ N & (X-NQ)D \end{bmatrix}$$

$Y_1 = (Y-MQ)D$ and $X_1 = (X-NQ)D$

$\therefore K = (Y-MQ)(X-NQ)^{-1}$

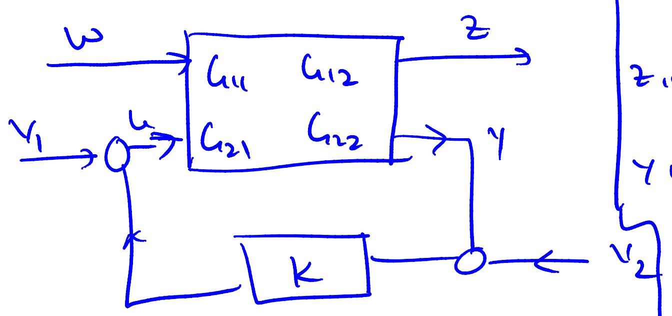
$\therefore$ Any stabilizing controller $K = Y_1 X_1^{-1}$ (being there(s)

can be written as $\underline{K = (Y-MQ)(X-NQ)^{-1}}$

where Q is stable.

Theorem: K is a stabilizing controller for the $G_{22}-K$ interconnection if and only if $K = (Y-MQ)(X-NQ)^{-1}$ for some Q stable and $\begin{bmatrix} \tilde X & -\tilde Y \\ -\tilde N & \tilde M \end{bmatrix}\begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I$ is a dcf of G_{22}.

Generalized plant for the MIMO case:



w is exogenous input

z is the regulated variable

y is the measured output

K is the controller.

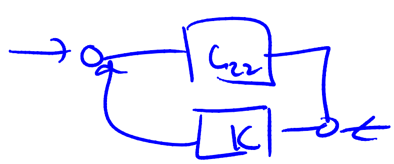
G_{22} is the actual plant

- $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ is the generalized plant

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$ (ignore $v_1 = v_2 = 0$)

$$y = G_{21} w + G_{22} u$$

G_{22} is the map from the control output to the output of the plant



→ Note that

Suppose $G = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a stabilizable and detectable realization of G. and

$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{bmatrix} G_{11} w + G_{12} u \\ G_{21} w + G_{22} u \end{bmatrix}$$

$$\begin{cases} \dot{x} = Ax + B\begin{bmatrix} w \\ u \end{bmatrix} ; \quad \begin{pmatrix} z \\ y \end{pmatrix} = \cancel{B}\, Cx + D\begin{bmatrix} w \\ u \end{bmatrix} \\ \quad = Ax + B_1 w + B_2 u \end{cases}$$

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} C_1 x + D_{11} w + D_{12} u \\ C_2 x + D_{21} w + D_{22} u \end{bmatrix}$$

$G = \quad \left[A \quad \middle| \quad B_1 \quad B_2 \right) \qquad \left[G_{11} \quad G_{12} \right)$

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

If above is the realization of G & what is the inherited realization of $G_{11}, G_{12}, G_{21}, G_{22}$?

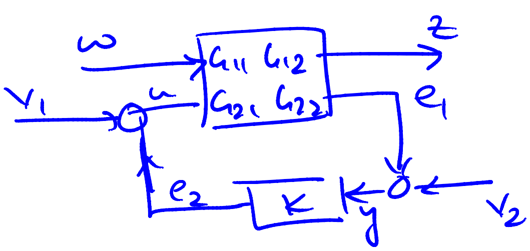
$$G_{11} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right]; \quad G_{12} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right]$$

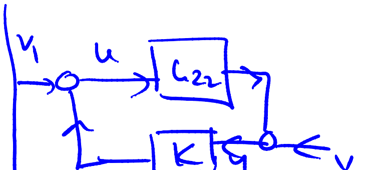
$$G_{21} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right]; \quad G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right].$$

These ~~real~~ inherited realization (even though $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a stabilizable and detectable realization of G) are not neccessarily stabilizable and detectable.

<u>Theorem</u>: The above $(G-K)$ interconnection is well-posed if and only

$$\det(I - D_{22} D_K) \neq 0.$$





$$\left(\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right); \quad \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

$$z = G_{11} w + G_{12}(v_1 + e_2)$$
$$e_1 = G_{21} w + G_{22}(v_1 + e_2)$$

$$T^{I} \; T \quad -G_{12} \quad 0 \; \Big) \Big(z \Big) \quad \Big(G_{11} \quad 0 \quad 0 \Big) \Big(w \; v_1 \ldots$$

$$e_1 = G_{21}w + G_{22}(v_1 + e_2)$$

$$\begin{bmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{bmatrix} \begin{pmatrix} z \\ u \\ y \end{pmatrix} = \begin{pmatrix} G_{11} & 0 & 0 \\ 0 & I & K \\ G_{21} & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix}$$

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & K \\ G_{21} & 0 & 0 \end{bmatrix}}_{H(G,K).} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = H(G,K) \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}.$$

→ The closed-loop interconnection is stable if and only if $\|H(G,K)\|_{H_\infty} < \infty$.