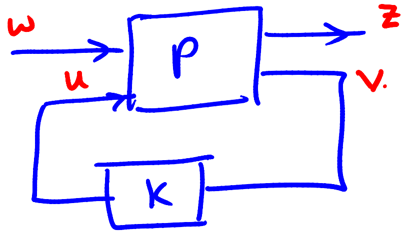
Lecture10

Tuesday, February 22, 2011
8:16 AM



$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = Kv$$

$$z = P_{11} w + P_{12} u$$
$$v = P_{21} w + P_{22} u$$
$$u = Kv$$
$$\Rightarrow v = P_{21} w + P_{22} K v$$
$$\Rightarrow (I - P_{22} K) v = P_{21} w$$
$$\Rightarrow v = (I - P_{22} K)^{-1} P_{21} w$$

$$z = P_{11} w + P_{12} u = P_{11} w + P_{12} K v$$
$$= P_{11} w + P_{12} [I - P_{22} K]^{-1} K P_{21} w$$

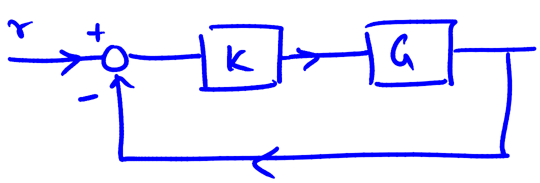
$$\therefore \quad z = \underbrace{\left[P_{11} + P_{12} [I - P_{22} K]^{-1} K P_{21} \right]}_{\text{closed-loop map}} w.$$

$$'' $$
$$F_\ell (P, K).$$

⊙ What is the set of closed-loop maps that are achievable via stabilizing controllers.?

$$\Phi := \{ F_\ell(P, K) : K \text{ is a stabilizing controller} \}$$

Parametrization of stabilizing controllers when the plant G is stable:



We know that $K = \frac{n_K}{d_K}$ [(n_K, d_K) being coprime] is stabilizing if and only if

$(n_u n_k + d_u d_k)$ with $\frac{n_u}{d_u}$ being coprime

has no roots in the RHP.

→ Suppose $K = \frac{n_k}{d_k}$ is stabilizing.

the we know that

$n_u n_k + d_u d_k$ has no rhp roots

$$n_u[n_k - d_u q] + d_u\{d_k + n_u q\}$$

where q is a parameter

$$= n_u n_k - \cancel{n_u d_u q} + d_u d_k + \cancel{d_u n_u q}$$

$$= n_u n_k + d_u d_k$$

$\therefore$ we have that $\frac{n_k - d_u q}{d_k + n_u q}$ is also a stabilizing controller.

→ Suppose $\frac{n_k'}{d_k'}$ is another stabilizing controller

then does $\frac{n_k'}{d_k'}$ admit a form $\frac{n_k - d_u q}{d_k + n_u q}$ for some polynomial q.

→ $$\frac{n_k'}{d_k'} = \frac{n_k - d_u q}{d_k + n_u q}$$

$$n_k' d_k + n_k' n_u q = n_k d_k' - d_u d_k' q$$

$$\Rightarrow \quad q\,[n_k' n_u + d_u d_k'] = n_k d_k' - n_k' d_k$$

$$q = \frac{n_k d_k' - n_k' d_k}{n_k' n_u + d_u d_k'}$$

q is a ratio of polynomials.

→ Let S be the set of all stable rational and proper transfer functions.

→ Two elements of S are said to be coprime if they do not have any common rhp zero.

if they do not have any common rhp zero.

→ **Theorem:** Given $G = \frac{N}{M}$; $K = \frac{Y}{X}$ that are coprime representations with N, M, Y, X in S

Then K is a stabilizing controller if and only if $NY + MX$ has no rhp zeros.

Example: $G = \frac{s+1}{s-1}$; $K = \frac{1}{s+2}$

Coprime representation over polynomials : $n_G = s+1$
$d_G = s-1$
$n_K = 1$
$d_K = s+2$

Consider: $G = \dfrac{\frac{s+1}{s+1}}{\frac{(s-1)}{(s+1)}}$; $N = \frac{s+1}{s+1}$; $M = \frac{s-1}{s+1}$

N and M are coprime over S

$K = \dfrac{\frac{1}{s+2}}{1}$; $Y = 1/s+2$; $X = 1$.

Corollary: Suppose K is a stabilizing controller with $G = \frac{N}{M}$ being a coprime representation over S then $\exists$ a coprime representation over S $\frac{Y}{X} = K$

Such that

$$\underline{MX + NY} = 1.$$

Proof: Suppose $K = \frac{Y_1}{X_1}$ is a coprime representation over S of K that is stabilizing then from the previous theorem

$R = (MX_1 + NY_1)$ has no roots in the RHP

$\therefore$ R^{-1} is stable.

Let $X = X_1R^{-1}$; $Y = Y_1R^{-1}$

then $\frac{Y}{X} = \frac{Y_1R^{-1}}{X_1R^{-1}} = \frac{Y_1}{X_1} = K$

$$MX + NY = MX_1R^{-1} + NY_1R^{-1}$$
$$= (MX_1 + NY_1)R^{-1} = RR^{-1} = 1$$

That concludes the proof.

Parametrization of all stabilizing Controllers

→ Suppose $K = Y/X$ is a stabilizing controller with $G = N/M$ and
$$MX + NY = 1$$

Note that, Q stable
$$M[X - NQ] + N[Y + MQ]$$
$$= MX + NY = 1.$$

$\therefore$ $(X - NQ)$ is stable ; $Y + MQ \in S$

$\therefore \frac{Y_1}{X_1} = \frac{Y + MQ}{X - NQ}$ is another stabilizing controller.

Suppose $\frac{Y_1}{X_1}$ is a stabilizing controller
Lets solve for $\frac{Y_1}{X_1}$

$$\frac{Y_1}{X_1} = \frac{Y + MQ}{X - NQ}$$
$$XY_1 - NY_1Q = X_1Y + X_1MQ$$
$$\Rightarrow XY_1 - X_1Y = X_1MQ + NY_1Q$$
$$\Rightarrow Q = \frac{XY_1 - X_1Y}{X_1M + NY_1}$$

As $\frac{Y_1}{X_1}$ is stabilizing $MX_1 + NY_1$ has no rhp zeros $\therefore$ Q is Stable

$\therefore$ any stabilizing controller $\frac{Y_1}{X_1}$ can be written as $\frac{Y + MQ}{X - NQ}$. where $\frac{Y}{X}$ satisfies
$$MX + NY = 1.$$

Summary: Suppose $G = \frac{N}{M}$ is a coprime representation over S with $N, M \in S$. Suppose Y and X are such that
$$MX + NY = 1$$ with Y and X in S then all stabilizing Controllers are given by the Set

Controllers are given by the set

$$\left\{ \frac{Y+MQ}{X-NQ} \; : \; Q \in S \right\}$$

Example: Lets consider the sensitivity transfer function

$$S = (I+GK)^{-1} \quad \text{for some stabilizing } K$$

$$W_pS = W_p \frac{1}{1+\frac{N}{M}\frac{(Y+MQ)}{(X-NQ)}}$$

$$= W_p \frac{M(X-NQ)}{MX - \cancel{MNQ} + NY + \cancel{NMQ}}$$

$$= W_p \frac{(MX-MNQ)}{MX+NY} = W_pMX - W_pMNQ.$$

$$\left\{ W_pS = \frac{W_p}{1+GK} \text{ such that } K \text{ is stabilizing} \right\}$$

$$= \{ W_pMX - W_pMNQ \; : \; Q \text{ is stable} \}$$

Suppose we need tracking of steps with zero steady state error

From final value theorem

$$e_{ss}(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\,S(s)Y(s)$$

$$= \lim_{s\to 0} s\,S(s)\frac{1}{s}$$

$$= S(0)$$

$S(0) = 0 \Rightarrow$ find a Q stable such that

$S = (MX - MNQ)$

$$(MX - MNQ)(0) = 0$$

$$\Leftrightarrow \quad (MX)(0) = (MNQ)(0)$$

Suppose we need a stabilizing controller K such that $\|W_pS\|_{H_\infty} \le \gamma$

$\Leftrightarrow$ Find Q stable such that

$$\| W_pMX - W_pMNQ \|_{H_\infty} \le \gamma$$

Suppose $\displaystyle \min_{\text{all stabilizing}} \|W_pS\|_{H_\infty} = \min_{Q \text{ stable}} \|W_pMX - W_pMNQ\|_{H_\infty}$

controller...

Suppose W_p is stable and minimum phase. Suppose N is stable and ~~not~~ minimum phase

$$W_p [M x - {}^{M}N Q]$$

$$\mathbf{0} \quad M x \overset{M}{=} N Q \Rightarrow Q = \frac{}{N} x$$

$$\gamma = \min_{Q \text{ stable.}} \| W_p M (x - N Q) \|_{H_\infty}$$

$$= \min \|\phi\|_{H_\infty}$$

s.t. $\phi = W_p M (x - N Q)$

and Q stable.

$$\phi - w_p M x = - w_p M N Q \Rightarrow Q = \frac{\phi - W_p M x}{W_p M N}$$

$$= \min \|\phi\|_{H_\infty}$$

s.t $\phi = W_p M x - W_p M N Q$

Q is stable

$$= \min \|\phi\|_{H_\infty}$$

$\phi = W_p M (x - N Q)$

$Q = -\dfrac{[\phi - W_p M x]}{[W_p M N]}$ Stable.

$\Rightarrow Q$ will be stable if for every unstable zero of MN, (lets assume simple zeros) $(\phi - W_p M x)$ will be zero too.

$$= \min_{\text{s.t}} \|\phi\|_{H_\infty}$$

$\phi = W_p M (x - N Q)$

$\phi(s_0) = (W_p M x)(s_0)$

$$= \min \|\phi\|_{H_\infty} \longrightarrow \|\phi\|_{H}$$

$\phi(s_0) = (W_p M x)(s_0).$