Robust Control: HW 9

Problem 1:

1. Prove that $\underline{\sigma} = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.

2. Prove that $\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$.

3. Give an example of a 2×2 matrix $A(\epsilon)$ that has stable eigenvalues that are constant independent of ϵ but $\bar{\sigma}(A(\epsilon)) \to \infty$ with $\epsilon \to \infty$.

4. Construct matrices $A(\epsilon)$, B, C, D (Note that B, C and D have to be constant matrices) such that if $G(s) = \left[\begin{array}{c|c} A(\epsilon) & B \\ \hline C & D \end{array}\right]$ then $\|G(s)\|_{\mathcal{H}_\infty} \to \infty$ with $\epsilon \to \infty$but the poles of G are independent of ϵ. Interpret this result.

Problem 2: Consider a spring-mass damper system whose dynamics is given by

$$\ddot{p} + \frac{c}{m}\dot{p} + \frac{k}{m}p = \frac{1}{m}u$$

where u is the control input. Assume that the displacement p is measured. Suppose that the mass m is within 1% of a nominal value $\bar{m}$, the stiffness k is within 1% of a nominal value $\bar{k}$, and the damping c is within 1% of a nominal value $\bar{c}$. Also assume that there is output multiplicative uncertainty of the form $(1 + W_m(s)\Delta_m)$ where $W_m(s)$ is stable and $\Delta(s)$ is stable with $\|\Delta\|_\infty \leq 1$. Noise n effects the measurement p. The reference trajectories to be tracked is captured by a stable prefilter W_r. The regulated variables are the error in tracking $r - p$, the control input u and the position p.

1. Cast the problem into a $G-K-\Delta$ framework with the G identified, the structure of Δ specified, the exogenous input vector and the regulated variable identified.

2. State the robust stability problem in terms of structured singular value.

3. State the robust performance problem in terms of structured singular value.

Problem 3: Consider a unity negative feedback system with $K(s) = \frac{1}{s}$ and a nominal plant model $\frac{s+1}{s^2+0.2s+5}$. Construct the smallest destabilizing $\Delta \in \mathcal{RH}_\infty$ in the sense of $\|\Delta\|_\infty$ for the following cases:

1. $P = P + \Delta$.
2. $P = P_0(1 + W\Delta)$ where $W(s) = \frac{0.2(s+10)}{s+50}$.
3. $P = \frac{N+\Delta_n}{M+\Delta_m}$, $N = \frac{2(s+1)}{(s+2)^2}$, $M = \frac{s^2+0.2s+5}{(s+2)^2}$ and $\Delta = [\Delta_n \;\; \Delta_m]$.

Problem 4: (Unstructured Perturbations)

1. (Additive Uncertainty) Let $\Pi = \{P + W_1 \Delta W_2 : \Delta \text{ is a stable transfer matrix}\}$. Suppose W_1 and W_2 are stable transfer matrices. Suppose K stabilizes P in a negative feedback interconnection. Show that the negative feedback interconnection of K and any plant in Π with $\|\Delta\|_\infty < 1$ is internally stable if and only if

$$\|W_2 K S_o W_1\|_\infty \leq 1$$

where $S_o := (I + PK)^{-1}$.

2. (Multiplicative Uncertainty) Let $\Pi = \{(I + W_1 \Delta W_2)P : \Delta \text{ is a stable transfer matrix}\}$. Suppose W_1 and W_2 are stable transfer matrices. Suppose K stabilizes P in a negative feedback interconnection. Show that the negative feedback interconnection of K and any plant in Π with $\|\Delta\|_\infty < 1$ is internally stable if and only if

$$\|W_2 T_o W_1\|_\infty \leq 1$$

where $T_o := I - S_o$.

3. Let $P = (I + \Delta W)P_0$ where Δ is stable with $\|\Delta\|_\infty < 1$. Also P and P_0 have the same number of unstable poles. Show that K robustly stabilizes P if and only if K stabilizes P_0 and

$$\|W P_0 K (I + P_0 K)^{-1}\|_\infty \leq 1.$$

Hint: Use the small gain theorem for unstructured uncertainty

4. Let

$$P_0 = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}.$$

(a) Suppose $P = P_0 + \Delta$ with Δ stable and $\|\Delta\|_\infty \leq \gamma$. Determine the smallest γ for robust stability.

(b) Let $\Delta = diag(k_1, k_2)$. Determine the stability region.

Problem 5: Consider the feedback system shown in Figure 1 where

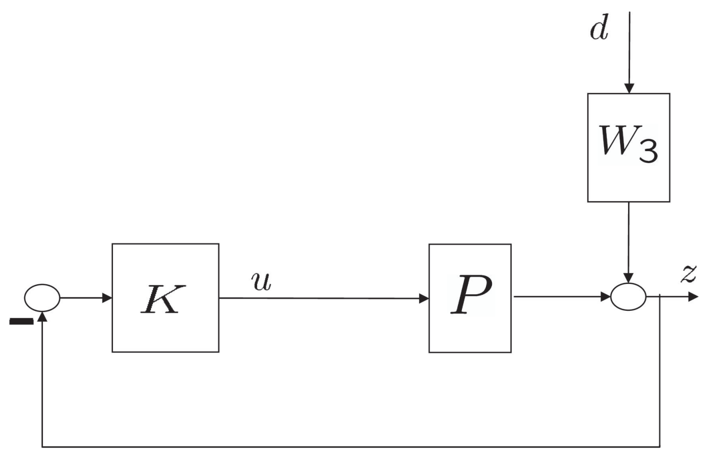

Figure 1:

$$P = P_0(1 + W_1\Delta_1) + W_2\Delta_2, \ \|\Delta_i\|_\infty < 1, \ i = 1, 2.$$

Suppose W_1 and W_2 are stable and P and P_0 have the same number if rhp poles.

1. Show that the interconnection is robustly stable if and only if K stabilizes P_0 and
$$\| \, |W_1 T| + |W_2 K S| \, \|_\infty \leq 1$$
where
$$S = \frac{1}{1 + P_0 K} \text{ and } T = \frac{P_0 K}{1 + P_0 K}$$

2. Show that the feedback system achieves robust performance (that is $\|T_{zd}\|_\infty \leq 1$) if and only if K stabilizes P_0 and
$$\| \, |W_3 S| + |W_1 T| + |W_2 K S| \, \|_\infty \leq 1$$