Homework 3
Robust Control

Problem 1[Weights on Sensitivity]

Consider the possible transfer function weights on the sensitivity closed-loop map given bu

$$w_{p1} = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \text{ and } w_{p2} = \frac{(s/M^{0.5} + \omega_B^*)^2}{(s + \omega_B^* A^{0.5})^2}.$$

1. Make an asymptotic plot of $\frac{1}{w_{p1}}$ and $\frac{1}{w_{p2}}$.
2. Comment on the tracking bandwidth, tracking error specification and the robustness imposed on the sensitivity by the two weights.
3. What is the slope of the bode plot after the tracking bandwidth for $\frac{1}{w_{p1}}$ and $\frac{1}{w_{p2}}$.

Problem 2 [Disturbance rejection]

We have seen that for the unity negative feedback configuration $e = SG_d d$ denote the tracking error with respect to the disturbance input. What would you suggest the weight w_p on the sensitivity transfer function be for good disturbance rejection. Assume that G_d is stable, minimum phase and bi-proper (that is G_d and G_d^{-1} both are proper) transfer function.

Problem 3[Stability Margins]

Let

$$M_S = \sup_\omega |S(j\omega)| \text{ and } M_T = \sup_\omega |T(j\omega)|.$$

Show that

1. $GM \geq \frac{M_S}{M_S - 1}$ and $PM \geq 2arcsin(\frac{1}{2M_S}) \geq \frac{1}{M_S}$
2. $GM \geq 1 + \frac{1}{M_T}$ and $PM \geq 2arcsin(\frac{1}{2M_T}) \geq \frac{1}{M_T}$
3. Show that the closest distance of the Nyquist plot of $L = GK$ from the $(-1, 0)$ point on the complex plane is given by M_S^{-1}.

where GM and PM denote the gain and phase margin respectively. Thus if M_S is small one has good gain and phase margins.

Problem 4[Loop Shaping] Let S be the sensitivity and L be the open-loop map GK.

1. We have seen that for good tracking we need $|S|$ to be small in the bandwidth where the tracking is desired. Show that

   (a) If $|S(j\omega)| \leq \epsilon_S$ then $|L(j\omega)| \geq \frac{1}{\epsilon_S} - 1$.

   (b) If $|L(j\omega)| \geq \epsilon_L > 1$ then $|S| \leq \frac{1}{\epsilon_L - 1}$.

   Thus small sensitivity implies large open-loop gain and large open-loop gain implies small sensitivity. Thus good tracking implies large open loop gain and good tracking is possible only when the open-loop gain is large.

2. For good noise rejection we need the complimentary transfer function T to be small. Show that

   (a) $|T(j\omega)| \leq \epsilon_T$ implies that $|L(j\omega)| \leq \frac{\epsilon_T}{1-\epsilon_T}$.

   (b) $|L(j\omega)| \leq \epsilon_L < 1$ implies that $|T(j\omega)| \leq \frac{\epsilon_L}{1-\epsilon_L}$.

   Thus good noise rejection implies small open loop gain and good noise rejection is possible only when the open-loop gain is large.

Classical control methods strives to design controllers K such that the open-loop gain L has the desired characteristics of large gain in the frequency region $[0, \omega_c]$ where tracking is desired and low gain where the noise rejection is desired. These designs also have to pay attention to stability requirements. Note that for stability, for typical plants, at the phase cross-over frequency ω_{180} the open-loop gain $|L(j\omega_{180}| < 1$. Thus this implies that the tracking bandwidth ω_c has to be less that ω_{180}. Good noise rejection is imposed by having $|L(j\omega)| \leq \epsilon_L$ in the region $\omega \geq \omega_T$. Thus $|L|$ has to roll-off from a high value at lower frequencies (for good tracking) to small values in the high frequency regions (for stability and good noise rejection).

- Show that the roll-off rate of $|L|$ cannot be better than $40db/decade$ (i.e. the magnitude part of the bode plot of L can have a slope more

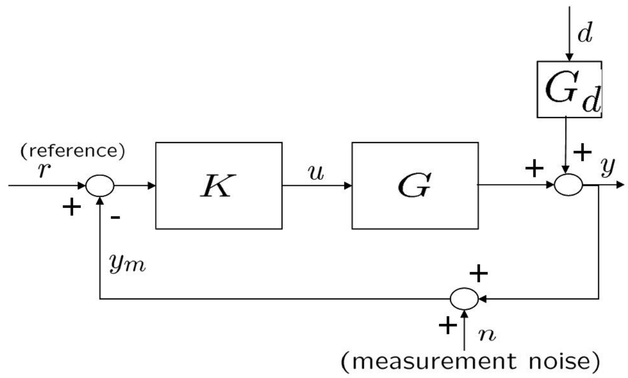

Figure 1:

negative than -2). This implies that, it is not possible to sharply let L transition from a large value to a small value which imposes limitations on performance.

In modern control design the specifications are directly imposed on closed loop maps, unlike classical designs that have to first infer the conditions that the open loop gain has to satisfy that would give good closed-loop performance. Also, the arguments for classical loop-shaping work typical plants where the bode plot arguments for stability hold. The modern designs do not have such restrictions.

Problem 5:[A mixed sensitivity design example]

Consider Figure 1 where the plant $G = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}$ and $G_d = \frac{100}{10s+1}$. The objectives are

1. Command tracking (that is tracking r).
2. Disturbance rejection
3. Meet control authority constraints.

To satisfy the command tracking and disturbance rejection constraints the weight on the sensitivity transfer function was chosen as

$$w_{p1}(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}$$

where

$$M = 1.5,\ \omega_B^* = 10,\ A = 10^{-4}.$$

The weight w_u was chosen as a constant 1. Thus the performance constraints are on

$$N = \begin{bmatrix} w_{p1} S \\ w_u K S \end{bmatrix}.$$

1. Form the generalized plant that captures the above performance constraint.

2. Use robust control or the μ toolbox of MATLAB to solve for the related $\mathcal{H}_\infty$ problem. Provide the optimal value of the solution.

3. Simulate the response of the closed-loop system for a step unit (i.e. when r is a step).

4. Simulate and comment on the response the closed-loop due to a disturbance that is a step.

5. Repeat all the steps when the weight on the sensitivity transfer function is given by

$$w_{p2}(s) = \frac{(s/M^{0.5} + \omega_B^*)^2}{(s + \omega_B^* A^{0.5})^2}$$

where

$$M = 1.5,\ \omega_B^* = 10,\ A = 10^{-4}.$$

Compare the results with the case when the weight on the sensitivity was w_{p1}.