Robust Control

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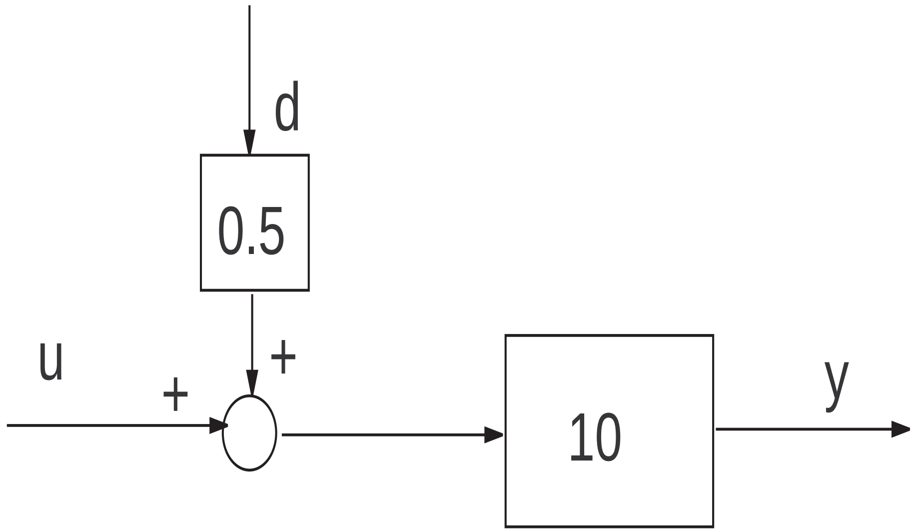
Preliminaries

Why Feedback?



- Why use feedback?

- Consider the problem described by the block diagram. It depicts a simplified version of cruise control problem in automobiles. The car transfer function between the fuel flow u and the speed y is given by a constant 10. This is the model when the road is flat.

- When the road has a gradient the discrepancy is modelled by adding $0.5d$ to the fuel flow (downhill is positive d.)

- Typically not much is known about the gradient of the road; thus the fuel flow has to be designed assuming a flat road. Thus if we want y to track a given reference r we determine the flow as $0.1r$ which results in $y = r$ when no gradient is present.

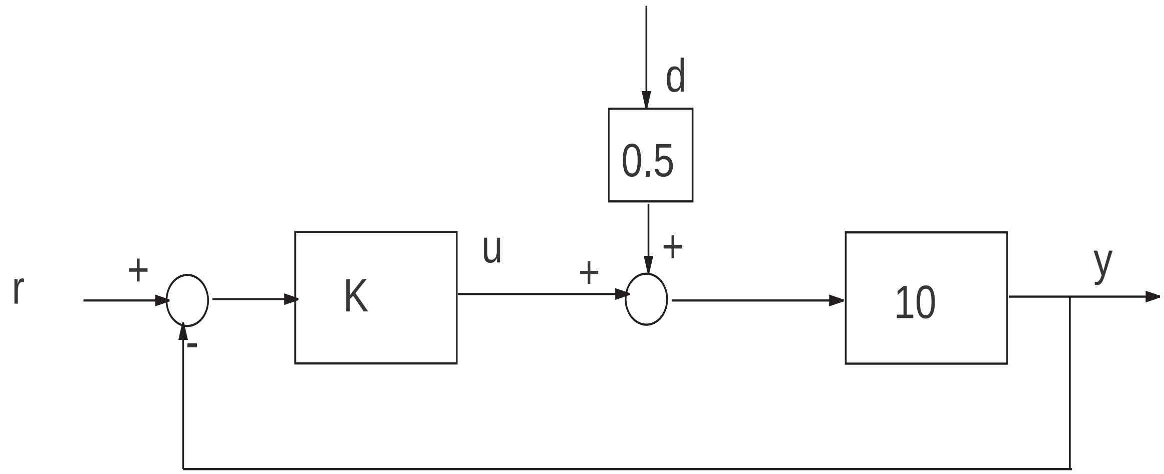
- The speed in the presence of the gradient y is given by

$$y = 10(0.5d + 0.1r) = r + 5d.$$

- Under no disturbance d the open loop controller $K = 0.1$ yield ideal tracking.

- However, under the disturbance (road gradient) the performance can be unsatisfactory.

Why Feedback?



- Consider the closed loop configuration shown in the figure.
- In the above setup

$$y = \frac{5}{1+10K}d + \frac{10K}{1+10K}r.$$

- It is clear that if K is chosen large then $\frac{5}{1+10K} \approx 0$ and $\frac{10K}{1+10K} \approx = 1$ and thus

$$y \approx r$$

making it insensitive to d. For example choose $K = 100$ then the contribution of d to y is $\frac{5}{1001}d$ which is much smaller than $5d$ that exists for the open loop case.

- Note that now even when $d = 0$, y is not equal to r as was the case with the open loop design.

- Evaluate the performance of the closed and the open loop when the plant model by the constant 10 is uncertain.

- Note that to implement the closed loop design one needs to sense the speed of the car so that it can be fed back. This involves sensors.

Furthermore the sensors are typically noisy and they do not yield the exact measurement of the car speed. Note that such effects of sensor noise are absent from the open loop design.

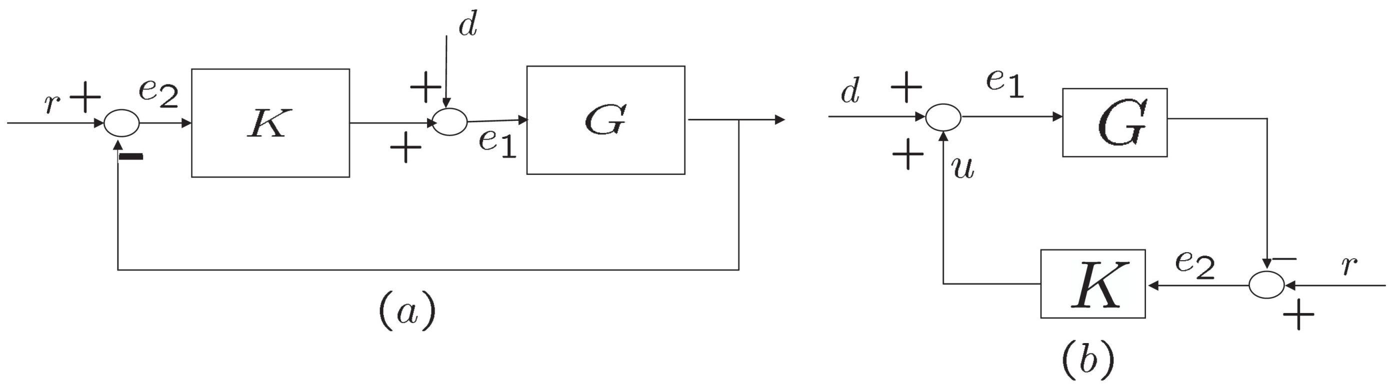
- Feedback controllers can stabilize unstable plants. However, bad controller design can lead to unstable closed loop systems even when the plant is stable.

Primary Reasons For Feedback Control

- **The primary reasons for feedback are**
  - ⋆ Model uncertainty
  - ⋆ Signal Uncertainty
  - ⋆ Stabilization

Single Input Single Output Interconnections: Stability

SISO Feedback Interconnection



Consider the unity negative feedback interconnection shown in Figure (a).

Definition 1. *The interconnection in Figure is said to be well posed if for any signals r and d there exist unique signals e_1 and e_2 that satisfy the loop-conditions implied by the interconnections.*

Note that

$$\begin{aligned} e_1 &= d + Ke_2 \\ e_2 &= r - Ge_1 \end{aligned}$$

That is

$$\begin{pmatrix} I & -K \\ G & I \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} d \\ r \end{pmatrix}$$

The following Theorem follows immediately:

Theorem 1. *The interconnection is well posed if and only if there exists some s_0 such that $G(s_0)K(s_0) + 1 \neq 0$.*

Let $G(s) = \frac{n_g}{d_g}$ and $K(s) = \frac{n_K}{d_K}$ where n_g, d_g and n_K, d_K are coprime polynomial pairs (no common factors).

It is evident that if the interconnection is well posed (we will assume this throughout unless mentioned otherwise) then

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} I & -K \\ G & I \end{pmatrix}^{-1} \begin{pmatrix} d \\ r \end{pmatrix}$$

and thus

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{I + GK} \begin{pmatrix} I & K \\ -G & I \end{pmatrix} \begin{pmatrix} d \\ r \end{pmatrix}$$

Definition 2. *The interconnection is stable if the map*

$$\begin{pmatrix} d \\ r \end{pmatrix} \mapsto \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

is bounded input bounded output.

The following theorem follows immediately

Theorem 2. *The interconnection is stable if and only if* $\frac{1}{1+GK}$, $\frac{G}{1+GK}$ *and* $\frac{K}{1+GK}$ *have no poles in the right half complex plane.*

Theorem 3. *The interconnection is stable if and only if the polynomial* $d_G d_K + n_G n_K$ *has no zeros in the right half complex plane.*

Proof:($\Leftarrow$) Suppose $d_G d_K + n_G n_K$ has no zeros in the right half complex plane. Note that

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \frac{1}{d_G d_K + n_G n_K} \begin{pmatrix} d_G d_K & d_G n_K \\ -d_K n_G & d_G d_K \end{pmatrix} \begin{pmatrix} d \\ r \end{pmatrix}$$

As the poles of all transfer functions are included in the zeros of the polynomial $d_G d_K + n_G n_K$ we have that all transfer functions are stable.

($\Rightarrow$) Suppose there is a s_0 with $Re(s_0) \geq 0$ and $(d_G d_K + n_G n_K)(s_0) = 0$. If the interconnection is stable then $\frac{1}{1+GK}$, $\frac{G}{1+GK}$ and $\frac{K}{1+GK}$ have all the poles in the strict left half plane. This implies that $\frac{d_G d_K}{d_G d_K + n_G n_K}$, $\frac{n_G d_K}{d_G d_K + n_G n_K}$ and $\frac{d_G n_K}{d_G d_K + n_G n_K}$ have no poles in the right half plane.

This implies that $d_G(s_0) d_K(s_0) = n_G(s_0) d_K(s_0) = d_G(s_0) n_K(s_0) = 0$ as the unstable pole at s_0 has to be cancelled by the respective numerator polynomials.

Note that as $d_G(s_0)d_K(s_0) = 0$ at least one of the terms $d_G(s_0)$ or $d_K(s_0)$ has to be zero. Lets assume that $d_G(s_0) \neq 0$. In this case $d_K(s_0) = 0$. We also have that $d_G(s_0)n_K(s_0) = 0$. As we have assumed that $d_G(s_0) \neq 0$ we have $n_K(s_0) = 0$. Thus we have that $d_K(s_0) = n_K(s_0) = 0$ which is a contradiction as we assumed that n_K and d_K are coprime polynomials (no common factors).

Similar conclusion can be reached if one assumes that $d_K(s_0) \neq 0$ in which case $n_G(s_0) = d_G(s_0) = 0$.

In case both $d_G(s_0) = d_K(s_0) = 0$ then as $n_k(s_0)n_G(s_0) + d_G(s_0)d_K(s_0) = 0$ it follows that $n_K(s_0)n_G(s_0) = 0$. This will again lead to the conclusion that either the plant or the controller representation is not coprime leading to a contradiction.

This proves the theorem. $\blacksquare$

Theorem 4. *The interconnection is stable if and only if*

1. $I + L$ with $L = GK$ has all zeros in the strict left half plane

2. There are no unstable pole-zero cancellations while forming the product $GK = \frac{n_G n_K}{d_G d_K}$. That is there no s_0 in the right half plane with $n_G(s_0)n_K(s_0) = d_G(s_0)d_K(s_0) = 0$.

Proof:$(\Rightarrow)$ Let the interconnection be stable. This implies that $n_G n_K + d_G d_K$ has no zeros in the right half plane. This implies that $I + L = \frac{n_G n_K + d_G d_K}{d_G d_K}$ has no zeros in the right half plane and thus (1) is satisfied. Also as $n_G(s_0)n_K(s_0) + d_G(s_0)d_k(s_0) \neq 0$ for all elements s_0 in the right half plane it follows that there can be no unstable pole zero cancellation in forming the product GK. This establishes (2).

$(\Leftarrow)$ Assume (1) and (2) are satisfied. Then it follows that $I + L = \frac{n_G n_K + d_G d_K}{d_G d_K}$ has no zeros in the right half plane. Suppose there exists a s_0 in the right half plane such that $n_G(s_0)n_K(s_0) + d_G(s_0)d_k(s_0) = 0$. Then this unstable pole has to be cancelled by the numerator i.e. $d_G(s_0)d_K(s_0) = 0$. This in turn would imply $n_K(s_0)n_G(s_0) = 0$ and an unstable pole-zero cancellation will ensue. This is a contradiction to (2).

This proves the theorem. $\blacksquare$.

Nyquist Plots

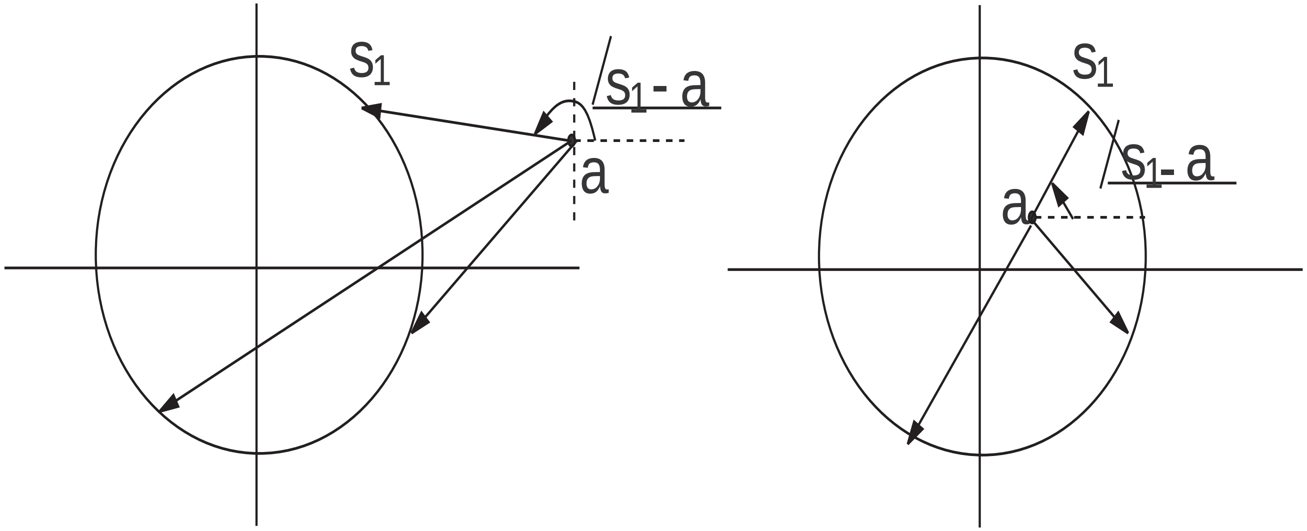
- Consider a transfer function $H(s)$. In the Nyquist plot of H, the imaginary part of $H(j\omega)$ is plotted against the real part of $H(j\omega)$.

Consider the transfer function $G(s) = s - a$. We will consider two cases

- What happens to the phase of $G(s)$ when s is traversed on a circle in the clockwise direction that does not contain a.

- What happens to the phase of $G(s)$ when s is traversed on a circle in the clockwise direction that contain a.

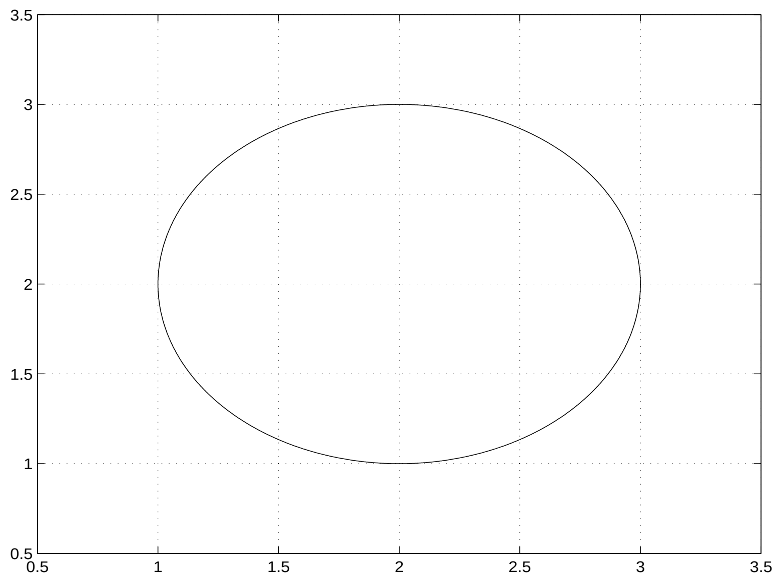
Argument Principle

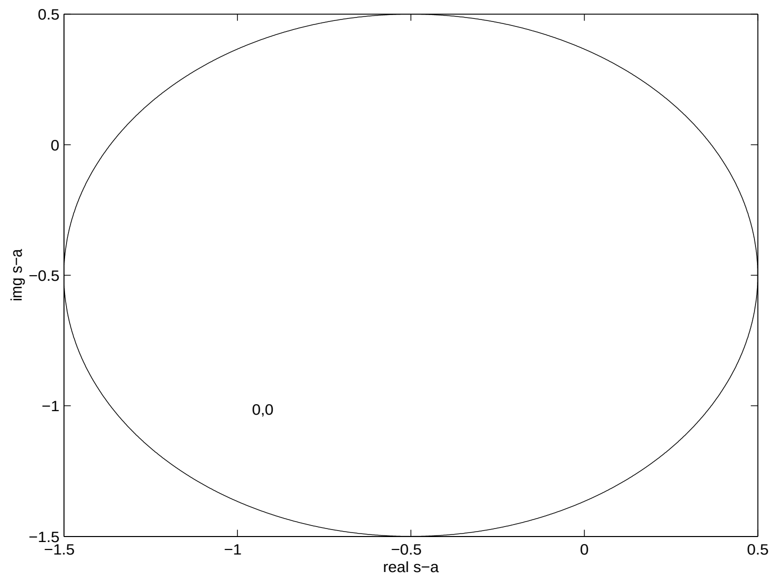


- In the case when a is is outside the contour (a circle in the figure) then $\angle H(s) = \angle(s-a)$ remains less than 360 deg as s is made to traverse the circle in the clockwise direction.

- In the case when a is is inside the contour (a circle in the figure) then $\angle H(s) = \angle(s-a)$ is equal to 360 deg as s is made to traverse the circle in the clockwise direction starting from s_1 and returning to s_1. As s is made traverse the circle in the clockwise direction the point $G(s)$ traverses around the origin in the clockwise direction.

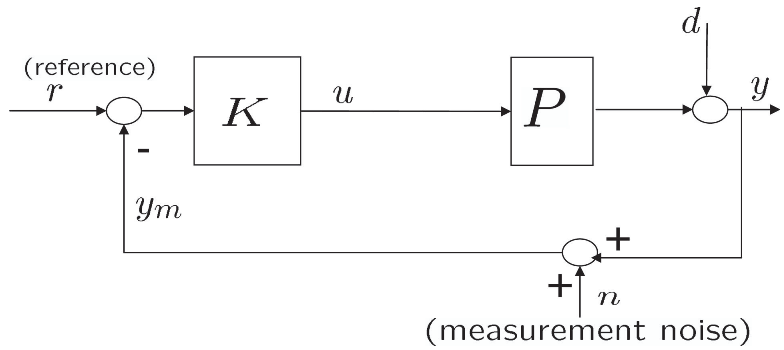
- Similarly the contour of $G(s)$ encircles the origin in the counterclockwise direction if $G(s)$ has a pole inside the countour that s traverses (note that $\angle(s-a) = -\angle(\frac{1}{s-a})$.





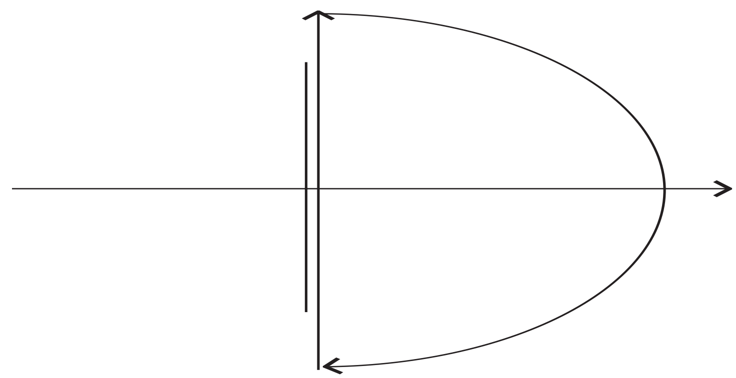
The Argument Principle: The contour map of a complex function $G(s)$ will encircle the origin $Z - P$ times in the clockwise direction when the contour itself is traversed in the clockwise direction where Z and P are the number of zeros and poles respectively of $G(s)$ that are inside the contour.

Nyquist Plot: Argument Principle used to Determine Stability



- The closed loop poles are the zeros of $1 + KG(s)$. Let the number of RHP zeros of $1 + KG$ be Z.

- The poles of $L := KG$ are same as the poles of $1 + KG = 1 + L$ which can be determined as K and G are known quantities. Let the number of right hand plane poles of L be P.



- Consider a contour that covers the entire RHP (called the Nyquist contour;shown above).

- The map of $1 + L$ will encircle the origin $N = Z - P$ times where P is a known quantity.

- This implies that L will encircle the origin $N = Z - P$ times.

- For stability we need $Z = 0$.

Theorem 5. *The interconnection is stable if and only if*

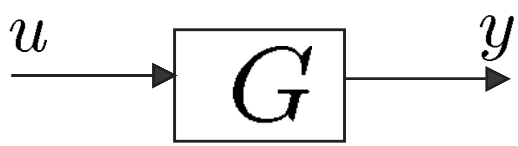
1. *The Nyquist plot of L encircles the -1 point in the counter-clockwise direction N number of times where N is equal to the poles of $L = GK$.*

2. *There are no unstable pole-zero cancellations while forming the product $GK = \frac{n_G n_K}{d_G d_K}$. That is there no s_0 in the right half plane with $n_G(s_0)n_K(s_0) = d_G(s_0)d_K(s_0) = 0.$*

Bode Plots

- Bode plot for a given frequency response function $H(j\omega)$ consists of two subplots
  - ⋆ the gain plot where the $\log_{10}|H(j\omega)|$ is plotted against $\log_{10}\omega$ for positive ω
  - ⋆ the phase $\angle H(j\omega)$ is plotted against $\log_{10}\omega$ for positive ω.

Bode Plots

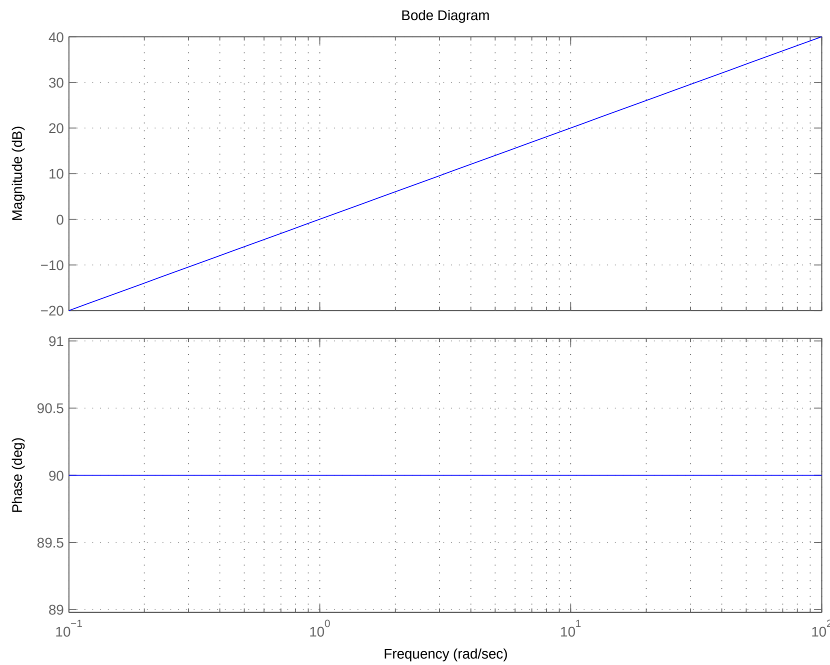


- Given a plant that is stable the bode plot can be obtained by following the following steps
  - ⋆ Give G an input $u(t) = A\sin(\omega t)$ and obtaining the steady state output $y(t)$. If the system is linear then $y(t)$ will be a sinusoid of the same frequency ω.
  - ⋆ Let $y(t) = y_\omega \sin(\omega t + \phi_\omega)$.
  - ⋆ Obtain the ratio $|\frac{y_\omega}{A}|$. This will be the magnitude of the frequency response $G(j\omega)$ at frequency ω.
  - ⋆ Set $\angle(G(j\omega)) = \phi_\omega$.
  - ⋆ Repeat the steps for various frequencies to obtain $G(j\omega)$.

Note that Spectrum Analyzers obtains the frequency response by essentially following the above steps and often provide $G(j\omega)$ as a complex number (Example: HP 3565 A).

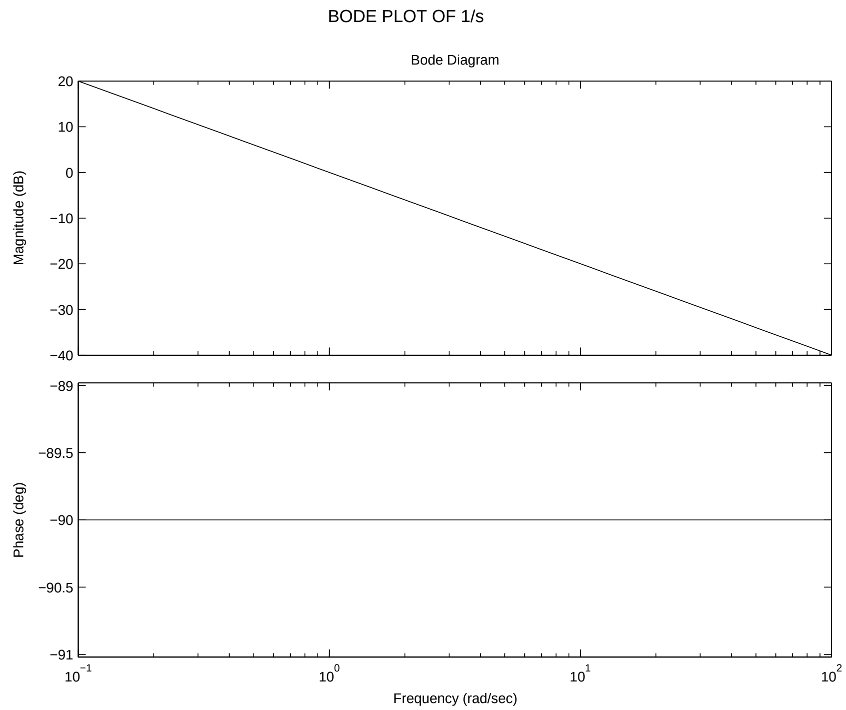
- If the plant is not stable then first it needs to be stabilized by some controller. The closed-loop system can now be used in the steps given above. In steady state all the internal signals in the plant controller interconnection will be sinusoidal with the same frequency as the frequency of the sinusoidal input to the closed loop system. The input and the output sinusoids of the plant G can be employed to determine $G(j\omega)$.

Bode plot of s



- $y = 20\log_{10}|G(j\omega)| = 20\log_{10}|j\omega| = 20\log_{10}|\omega| = 20x,$
- $\angle(G(j\omega)) = 90\ deg.$

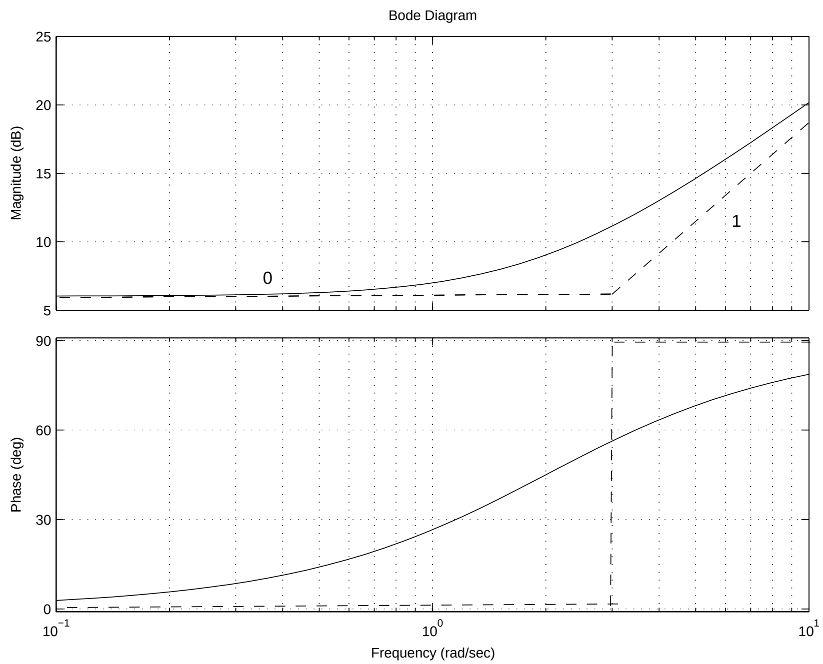
Bode plot Contd: plot of $1/s$



- $y = 20\log_{10}|G(j\omega)| = -20\log_{10}|j\omega| = -20\log_{10}|\omega| = -20x$
- $\angle(G(j\omega)) = -90\ deg.$

Bode plot Contd: plot of $s+2$

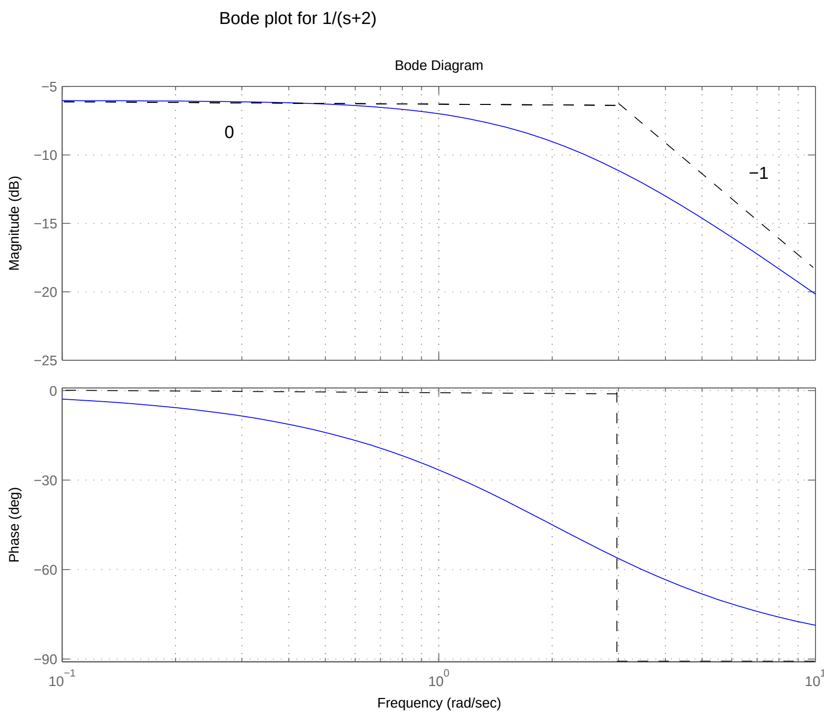
BODE PLOT OF s+2



Asymptotes

$$G(j\omega) = j\omega + 2 \begin{array}{ll} = & 2 \text{ if } |\omega| \leq 2 \\ = & j\omega \text{ if } |\omega| > 2. \end{array}$$

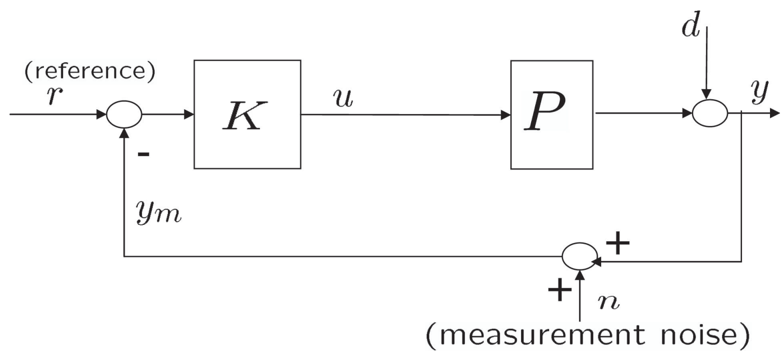
Bode plot of $1/(s+2)$



Asymptotes

$$G(j\omega) = \frac{1}{j\omega+2} \quad = \quad \frac{1}{2} \text{ if } |\omega| \leq 2$$
$$= \quad \frac{1}{j\omega} \text{ if } |\omega| > 2.$$

Bode's Criterion For Stability



Typical Case

- Let K be a positive scalar constant. A typical case is that the closed loop poles are all in the LHP for small enough K.

- As K is increased at least one of the closed-loop poles migrates into the RHP . The value of K when atleast one of the poles is on the imaginary axis is when KG is neutrally stable.

At this value of $K = K_n$

$$1 + K_n G(j\omega_{180}) = 0 \text{ and}$$

$$|K_n G(j\omega_{180})| = 1 \text{ and } \angle(K_n G(j\omega_{180}) = \angle(G(j\omega_{180}) = -180.$$

Note that ω_{180} is determined by G alone.

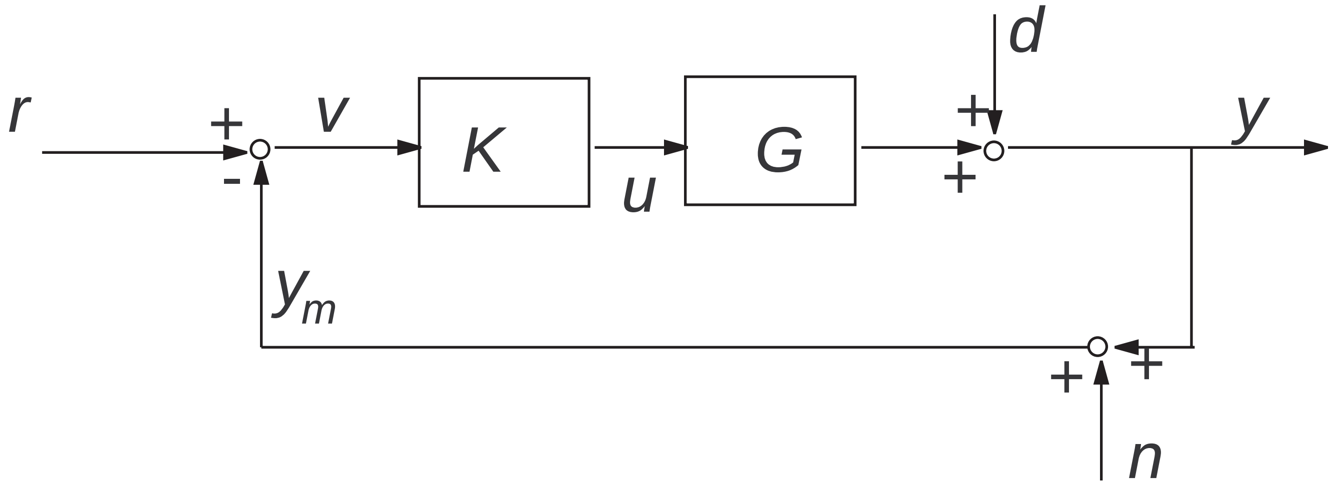
- Any value of K less than the neutral value leads to a stable closed loop system.
- This leads to the following conclusions: For all values of K that lead to stable closed loop maps $K < K_n$ which is true if and only if $|KG(j\omega_{180})| < |K_n G(j\omega_{180})| = 1$.
- Thus the rule in this case is that K leads to a stable closed loop map if

$$|KG(j\omega_{180})| < 1 \text{ where } \omega_{180} \text{ is defined by } G(j\omega_{180}) = -180.$$

Assumption is that $|G(j\omega)| = 1$ for a unique value of ω.

- Note that $G(j\omega)$ is the frequency response of the system.

Gain Cross Over Frequency

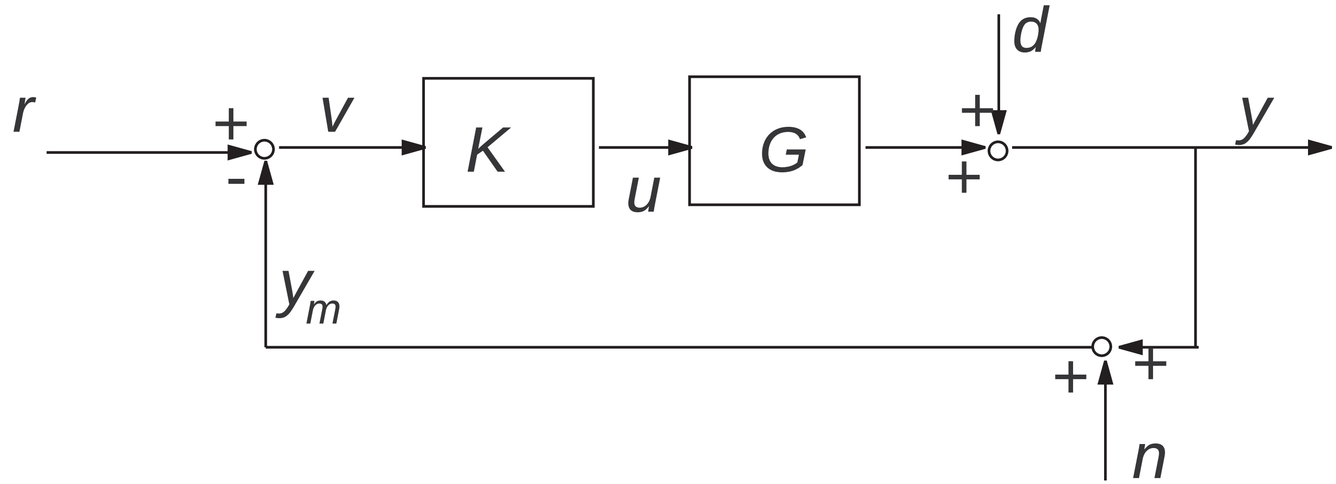


Definition 3. *Gain crossover frequency for the unity feedback configuration shown is defined to be the frequency* ω_c *which satisfies*

$$L(j\omega_c) = 1$$

where $L := GK$.

Phase Cross Over Frequency



Definition 4. *Phase crossover frequency for the unity feedback configuration shown is defined to be the frequency* ω_{180} *which satisfies*

$$\angle(L(j\omega_{180})) = -180$$

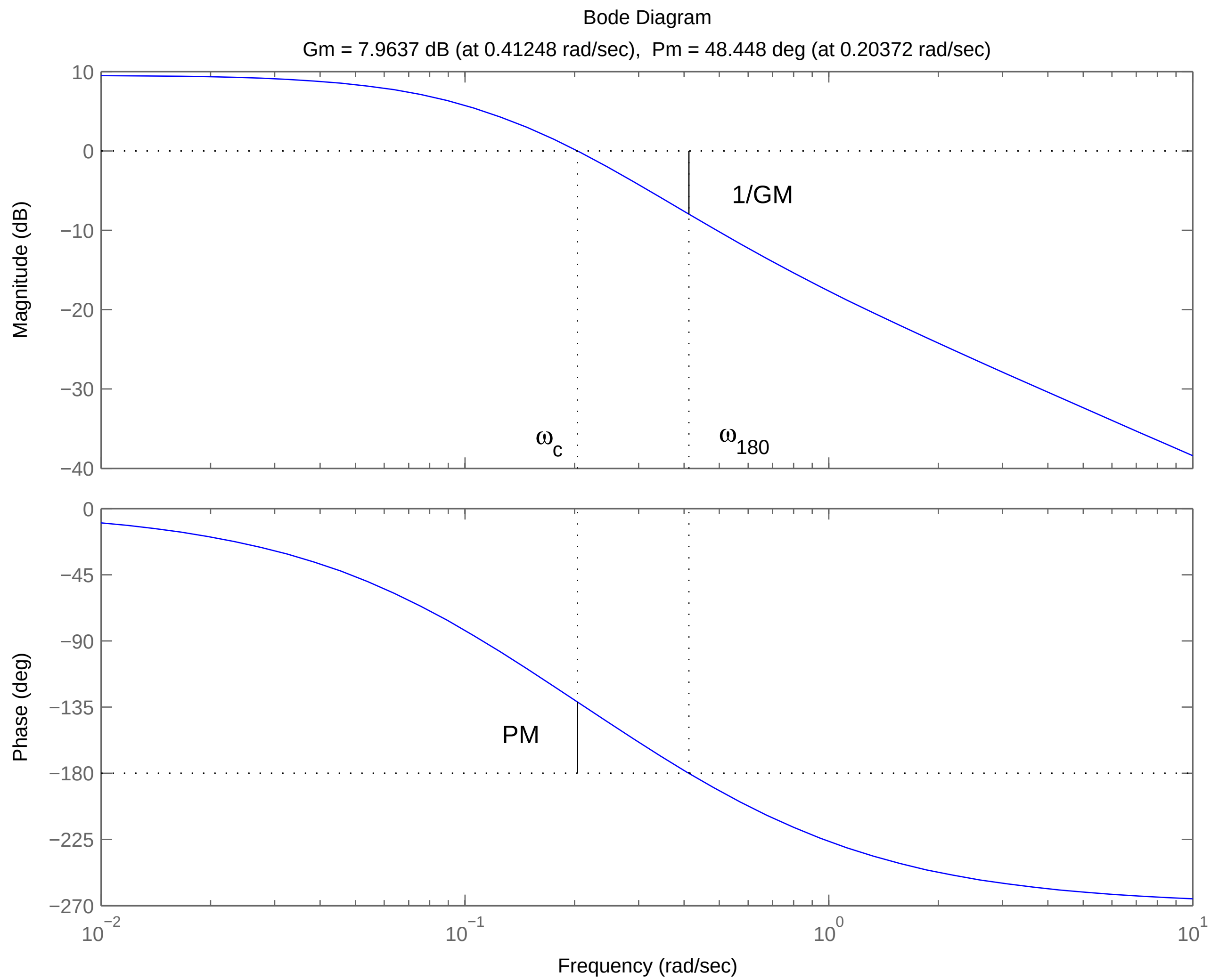
where $L := GK$.

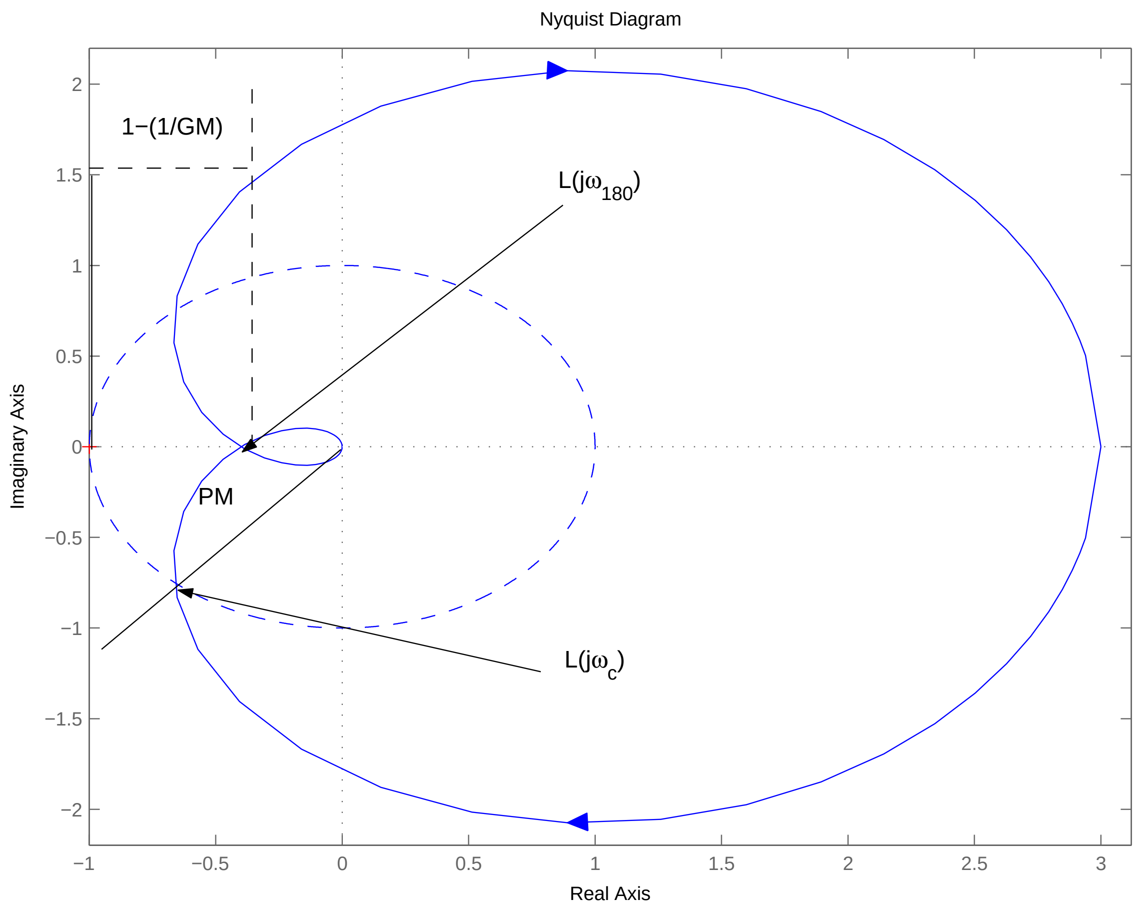
Stability Margins

- (*Gain Margin*) The factor by which the gain can be raised before instability occurs. This is given by
$$GM := |\frac{1}{L(j\omega_{180})}|$$
where ω_{180} is the *phase crossover frequency*. Clearly the the closed loop system is unstable if $GM < 1$. Typically a $GM > 2$ is desired.

- (*Phase Margin*) The phase that can be added at the gain crossover ω_c frequency before instability occurs. That is
$$PM := \angle(L(j\omega_c)) + 180$$
where ω_c is the *gain crossover frequency*. The closed loop system is unstable if GM is negative.

Stability Margins On the Nyquist Plot





- Phase and Gain margins for $L = -0.12\frac{(s-0.5)}{(s+0.1)(s+0.2)}$ on the Nyquist plot. Note that $P = 0$ and thus N has to be zero for stability.

Gain-Phase Relationship For Minimum Phase Systems

Suppose G is a LTI system that is such that $G(s)$ is analytic in the RHP (that is it is stable) and is minimum phase (that is it has no time delays or RHP zeros). Then the following identity holds

$$\angle G(j\omega_0) = \int_{-\infty}^{\infty} \underbrace{\frac{d\ln|G(j\omega)|}{d\ln\omega}}_{N(\omega)} \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right| \frac{1}{\omega} d\omega.$$

Thus the phase for such plants is completely determined by its gain $|G(j\omega)|$. Also, any other system which has the same gain as $|G(j\omega)|$ has <u>at least</u> as much phase as $\angle G(j\omega)$. That is why the system G is termed *minimum phase*.

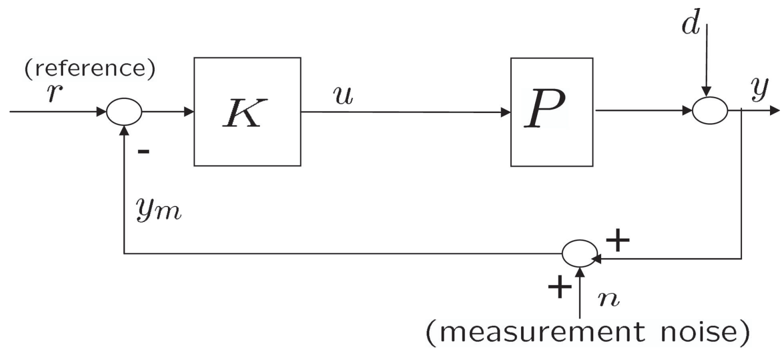
- It is clear that $\ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|$ takes large values near $\omega=\omega_0$ and thus $N(\omega)$ can be approximated by $N(\omega_0)$. Thus

$$\angle G(j\omega_0) \approx \int_{-\infty}^{\infty} N(\omega_0) \ln \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right| \frac{1}{\omega} d\omega = \frac{\pi}{2} N(\omega_0).$$

- Note that $N(\omega)$ is the slope of magnitude in the bode plot (that is $N(\omega) = \frac{d \ln |G(j\omega)|}{d \ln \omega}$ and in the bode plot $\log_{10} |G(j\omega)|$ is plotted against $\log_{10} \omega$).

Single Input Single Output Systems: Performance Measures

Unity Negative Feedback Configuration



- $y_m = y + n$ where y_m is the measured signal which is typically corrupted by noise n.

- $e = y - r$ where e is the error signal. Note that $e \neq y_m - r$ as is done in most treatments. $v = y_m - r$ is the input to the controller. The error signal is the difference between to be what is desired (r) and what the actual output is (that is y).

- u is the controller output

- $G_d d$ is a disturbance (typically has low frequency content).
- G is the plant
- K is the controller.

Important Closed Loop Transfer Functions

- $y = Gu + G_d d,\ u = K(r - y_m),\ y_m = y + n.$
- This implies that $y = GK(r - y_m) + G_d d = GKr - GKy - GKn + G_d d$
- Thus $(I + GK)y = GKr - GKn + G_d d.$
- Thus the output y is given by

$$y = \underbrace{(I + GK)^{-1}GK}_{T} r - \underbrace{(I + GK)^{-1}GK}_{T} n + \underbrace{(I + GK)^{-1}}_{S} G_d d.$$

- We have defined two important closed loop transfer functions
  - ⋆ Sensitivity transfer function $S = (I + GK)^{-1}$
  - ⋆ Complimentary transfer function $T = (I + GK)^{-1}GK.$

- ⋆ Note that $S + T = (I + GK)^{-1}(I + GK) = I$.

- Note that the error $e = y - r = (T - I)r - Tn + SG_d d$ Thus

$$e = -Sr - Tn + SG_d d.$$

Note that we have shown that $S + T = 1$.

It is worthwhile remembering that

- The sensitivity transfer function S is the map between the reference and the error. Thus small sensitivity S would imply good tracking.

- Small sensitivity S would imply good disturbance rejection.

- The complimentary transfer function is the map between the noise n and the error. Thus small complimentary sensitivity T would imply good noise rejection. Note that the noise n is absent in the open loop designs and thus closed-loop designs should be careful to minimize the effects of n typically caused by the sensor (otherwise the closed-loop can yield worse performance than the open-loop).

Remember: Minimize S for good tracking and good disturbance rejection, minimize T for good noise rejection.

- We have shown that $S + T = 1$. Thus it is clear that it is not possible to achieve small S and small T in the same frequency region.

- The reference trajectories to be tracked have low frequency content.

- The noise n effects only in the high bandwidth region (in the low bandwidth region as the noise is random there is time to average out the effect of noise).

- Thus S needs to be low in the low frequency region.

- T needs to be low in the high frequency region.

- Thus a tradeoff can be made between S and T as the objectives on S and T are in different frequency regions.

Shaping Closed Loop Transfer Functions

Typical Requirements on Sensitivity Transfer Function S.

- Minimum bandwidth frequency ω_B^* defined as the frequency where $S(j\omega)$ crosses 0.707 from below.

- $S(j\omega)$ not to exceed certain prespecified values at given frequencies $\omega = \omega_1, \ldots, \omega_n$ (maximum tracking error requirement at certain frequencies).

- S is to have a maximum peak magnitude M (robustness requirement as we will see later).

Mathematically the requirements can be captured by choosing and appropriate upper bound $w_p(j\omega)$ such that

$$|S(j\omega)| \leq \frac{1}{|w_p(j\omega)|} \forall \omega.$$

The above condition holds if and only if

$$|S(j\omega) w_p(j\omega)| \leq 1 \forall \omega$$

which holds if and only if

$$\sup_{\omega} |S(j\omega) w_p(j\omega)| \leq 1.$$

For any function $f(s)$ analytic in the RHP the $\mathcal{H}_\infty$ norm is defined as

$$\|f\|_{\mathcal{H}_\infty} = \sup_{\omega} |f(j\omega)|.$$

Thus the specifications on the sensitivity transfer function S takes the form

$$\|w_p S\|_{\mathcal{H}_\infty} \leq 1.$$

Weight Selection on S

Suppose the weight needs to capture the following specifications

- $\|S\|_{\mathcal{H}_\infty} \leq M_p$.
- $|S(j\omega)| \leq m_p$ for $\omega \leq \omega_p$.

Let $w_p(s) = \frac{s/M_p + \omega_p}{s + \omega_p m_p}$. Then

$$\|S w_p\|_{\mathcal{H}_\infty} \leq 1$$

imposes all the needed conditions.

Shaping Closed Loop Transfer Functions

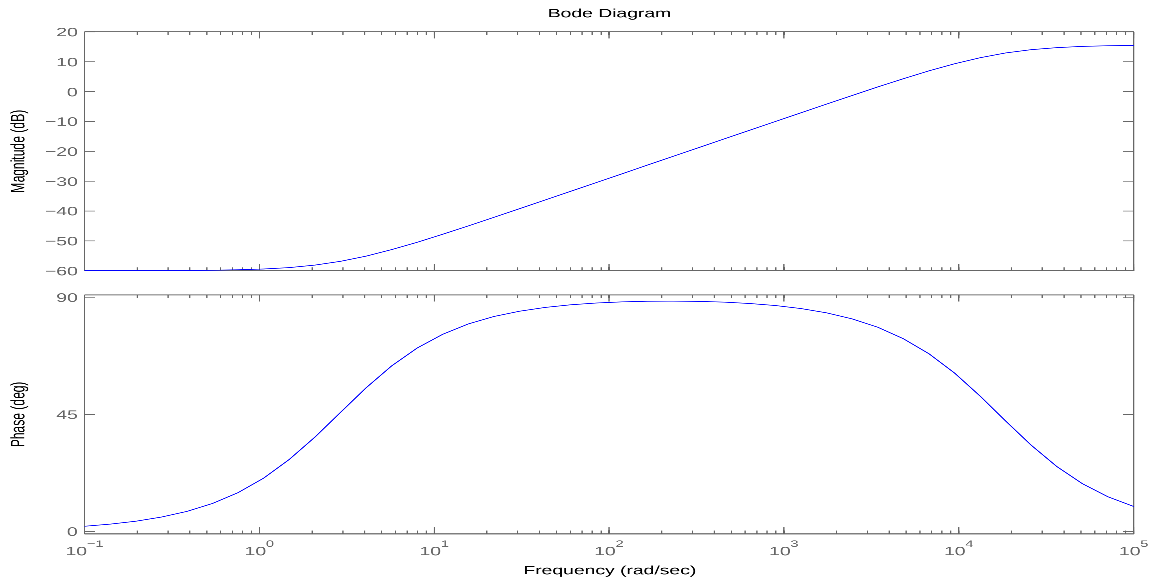
Typical Requirements on Sensitivity Transfer Function S.

- Minimum bandwidth frequency ω_B^* defined as the frequency where $S(j\omega)$ crosses 0.707 from below.

- $S(j\omega)$ not to exceed certain prespecified values at given frequencies $\omega = \omega_1, \ldots, \omega_n$ (maximum tracking error requirement at certain frequencies).

- S is to have a maximum peak magnitude M (robustness requirement).

Weight Selection For S



For example with $M_p = 6$, $m_p = 1e - 3$ and $\omega_P = 2827$ ($\omega_P = 2\pi f$ where the bandwidth is $f = 450Hz$.) we have

$$w_p = \frac{s/M_p + \omega_p}{s + \omega_p m_p} = \frac{0.1667s + 2827}{s + 2.827}.$$

The bode plot of $\frac{1}{w_p}$ is shown.

Shaping Closed Loop Transfer Functions

Typical Requirements on Complimentary Sensitivity Transfer Function T. Note that the weight on T should ensure that T is small at high frequencies.

- $|T(j\omega)| < 1/A_\ell$ for all $\omega < \omega_T - \Delta\omega$
- $|T(j\omega)| < A_h$ for all $\omega > \omega_T + \Delta\omega$

where typically $1/A_\ell \approx 1$ and thus does not conflict with the sensitivity weighting, A_h is small forcing T to be small in the high frequency region. A typical weighting function has the form

$$w_T = \frac{s + (1/A_\ell)\omega_T}{A_h s + \omega_T}.$$

The specifications on T can be achieved by imposing

$$|T(j\omega)| \le \frac{1}{|w_T(j\omega)|} \quad \text{for all } \omega$$

which holds if and only if

$$\|w_T T\|_{\mathcal{H}_\infty} \le 1$$

Typical Requirements on KS. The weight on KS is to restrict the magnitude of the control signal $u = KS(r - G_d d)$. Thus we need

$$|KS(j\omega)| \le \frac{1}{|w_u(j\omega)|}$$

which is satisfied if and only if

$$\|w_u KS\|_{\mathcal{H}_\infty} \le 1$$

Thus the requirements on the closed loop maps translate into the following conditions

- $\|w_p S\|_{\mathcal{H}_\infty} \leq 1$
- $\|w_T T\|_{\mathcal{H}_\infty} \leq 1$
- $\|w_u K S\|_{\mathcal{H}_\infty} \leq 1$

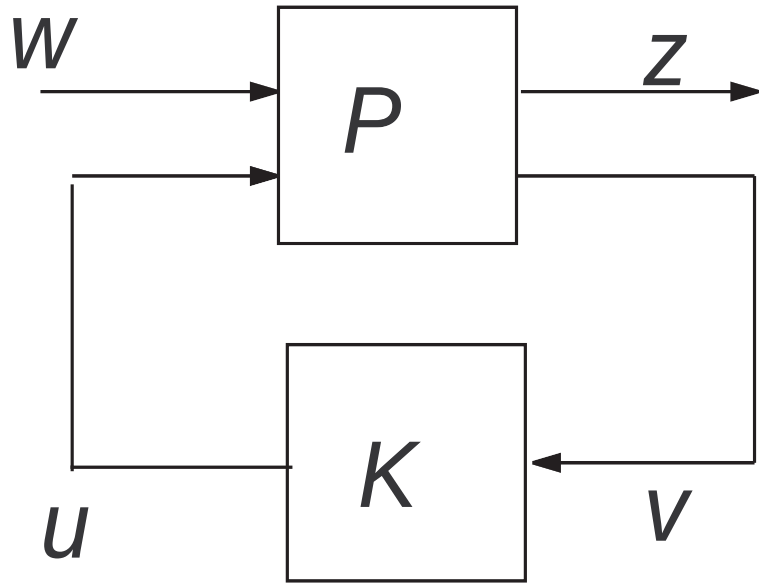
Note that the search of a controller that satisfies the above constraints is not what the standard $\mathcal{H}_\infty$ software solves. Instead the problem of finding a controller to satisfy the stacked constraint

$$\left\| \begin{matrix} w_p S \\ w_T T \\ w_u K S \end{matrix} \right\|_{\mathcal{H}_\infty} \leq 1$$

is solved where the $\mathcal{H}_\infty$ norm for a vector valued transfer function $f : C \rightarrow C^n$ is defined as

$$\|f\|_{\mathcal{H}_\infty} := \sup_{\omega} \overline{\sigma}(f(j\omega)).$$

Generalized Plant: The LFT Framework



- w : exogenous variables. This consists of all external signals including the reference signal.

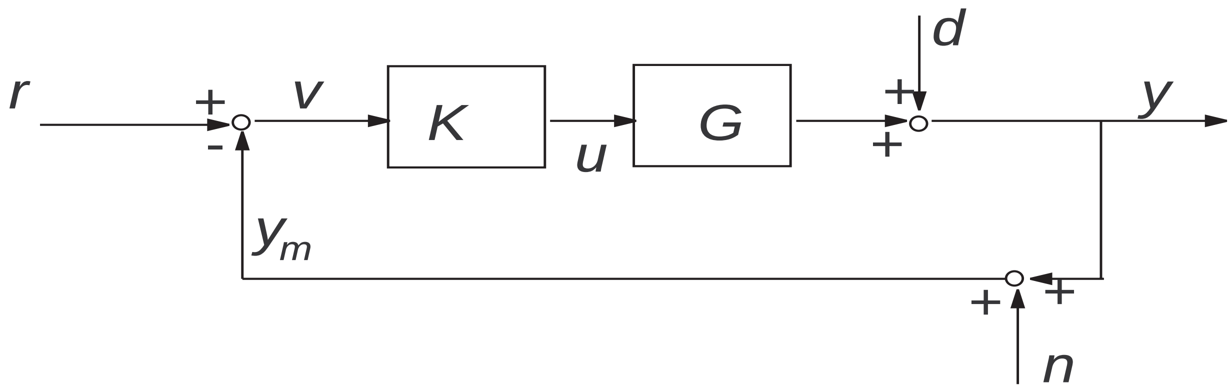
- z : regulated variables. These are the signals which have to be controlled.

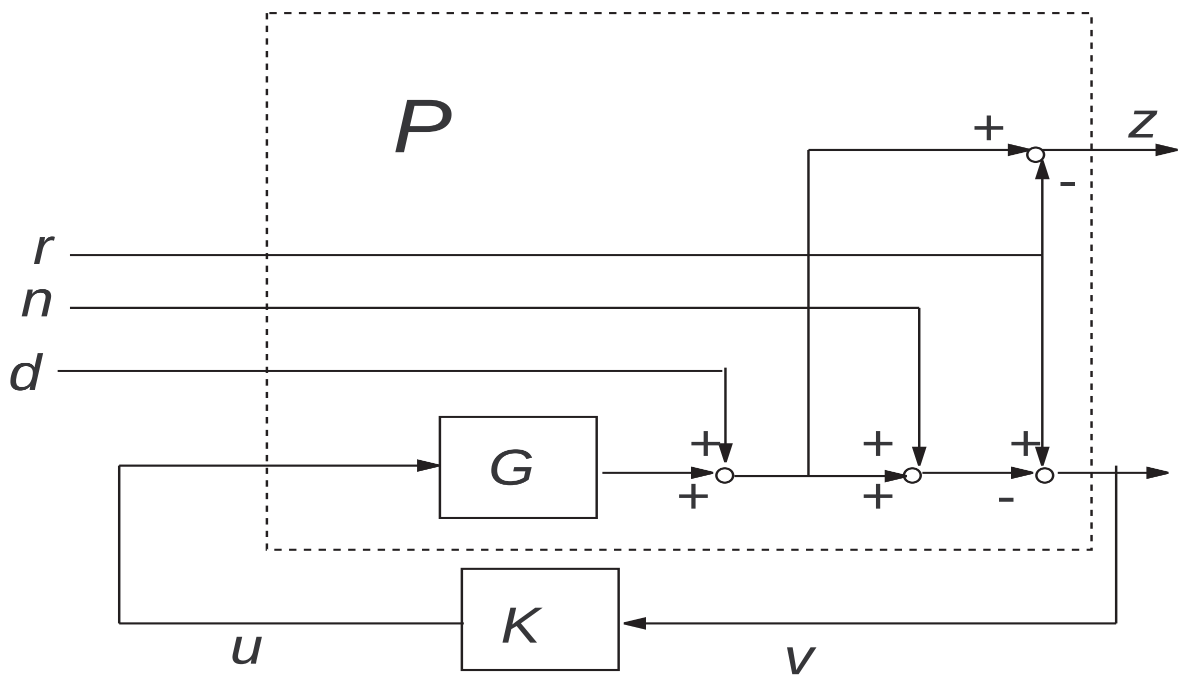
For example the error signal, the control signals.

- v : measured variables. These consist of the inputs *to* the controller. Usually the sensor output is fed to the controller.

- u : control variable. This is the output of the controller.

Generalized Plant: Example


d
r
+
v
K
G
+
y
-
u
+
y_m
+
+
n

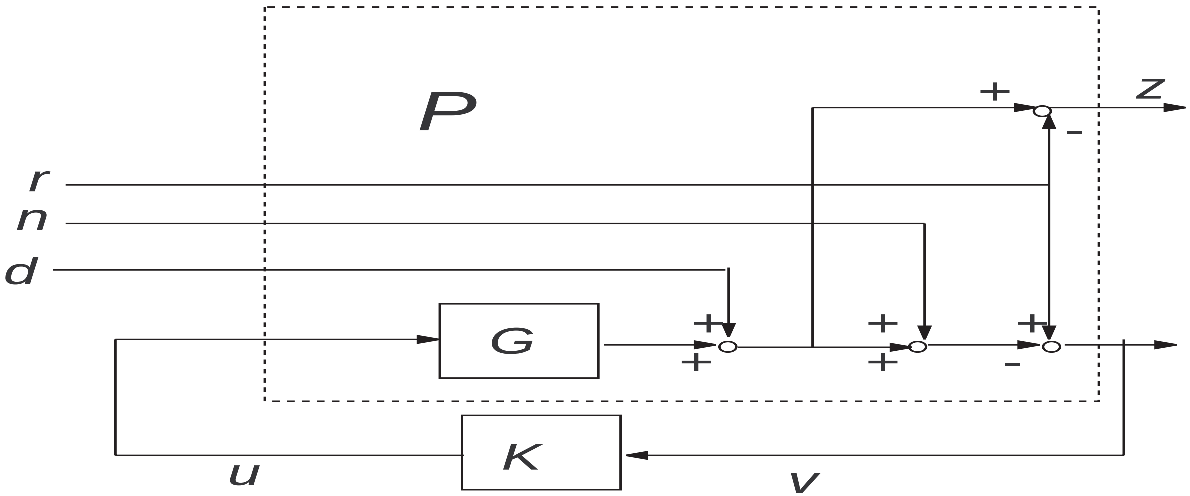

P
+
z
-
r
n
d
G
+
+
+
+
+
-
u
K
v

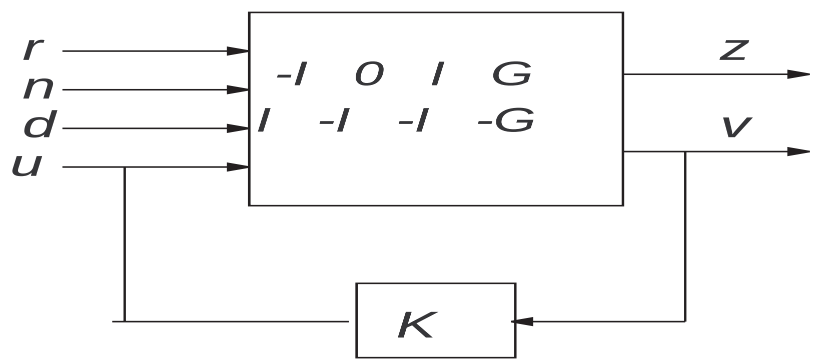
- $w = [r\ n\ d]'$,

- $z = y - r = Gu + d - r = [-I\ 0\ I\ G]\begin{bmatrix} w \\ u \end{bmatrix}$

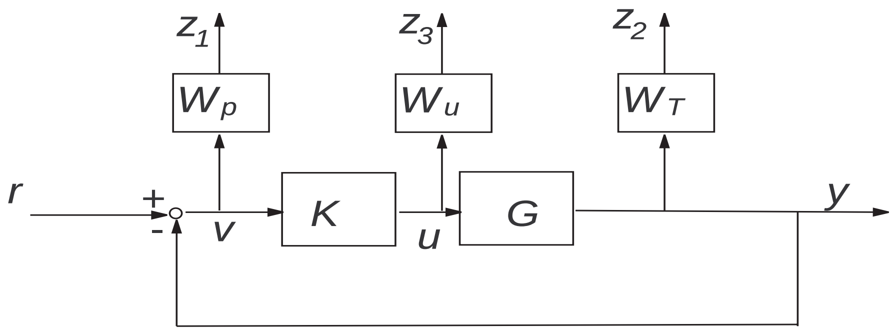
- $v = r - y_m = r - y - n = r - Gu - d - n = [I\ \ -I\ \ -I\ \ -G]\begin{bmatrix} w \\ u \end{bmatrix}$

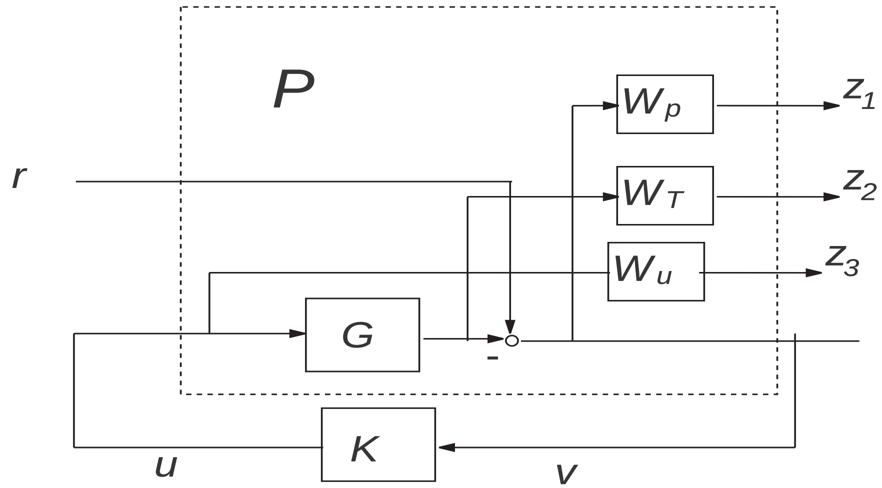
Generalized Plant: Example Contd.





Generalized Plant For The Stacked Problem





Generalized Plant For The Stacked Problem

- The transfer function between r and z_1 is $W_P S$.
- The transfer function between r and z_2 is $W_T T$.
- The transfer function between r and z_3 is $W_u K S$.

Thus the above setup describes the performance objectives.

The regulated outputs are given by

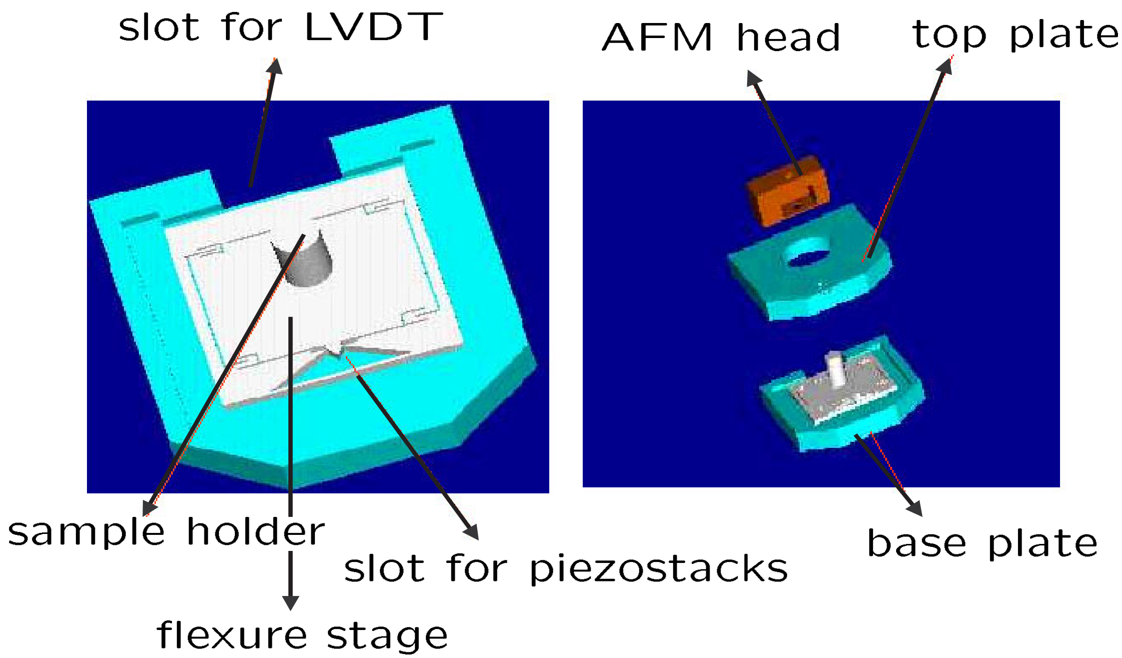
$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} W_p(r - Gu) \\ W_T G u \\ W_u u \end{bmatrix},$$

and the generalized plant, P is described by

$$\begin{bmatrix} z \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \begin{bmatrix} W_p & -W_pG \\ 0 & W_TG \\ 0 & W_u \end{bmatrix} \\ \begin{bmatrix} I & -G \end{bmatrix} \end{bmatrix}}_{=P} \begin{bmatrix} r \\ u \end{bmatrix}.$$

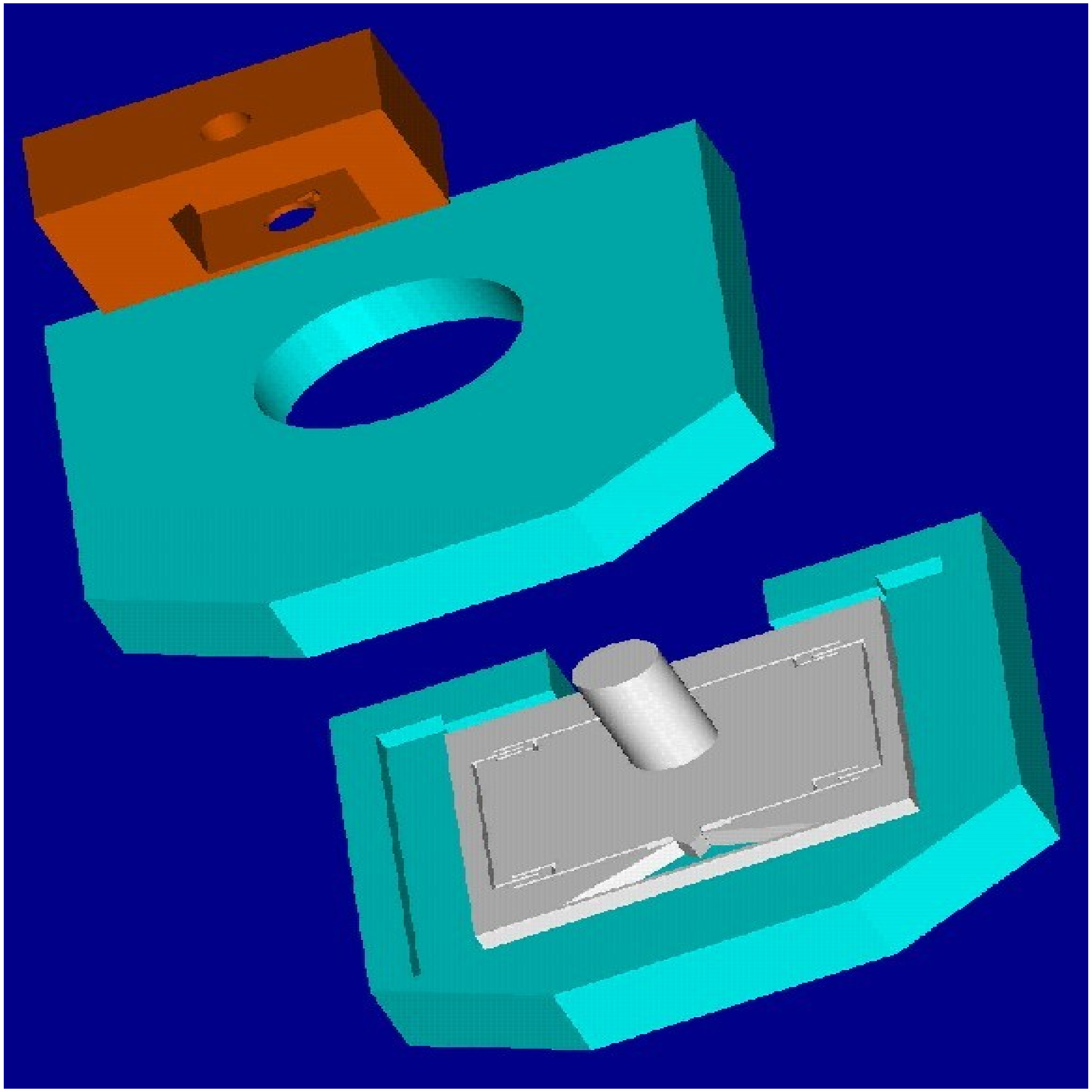
Nanopositioning: A Quick Introduction to $\mathcal{H}_\infty$ Control Design

Case Study: Nanopositioning

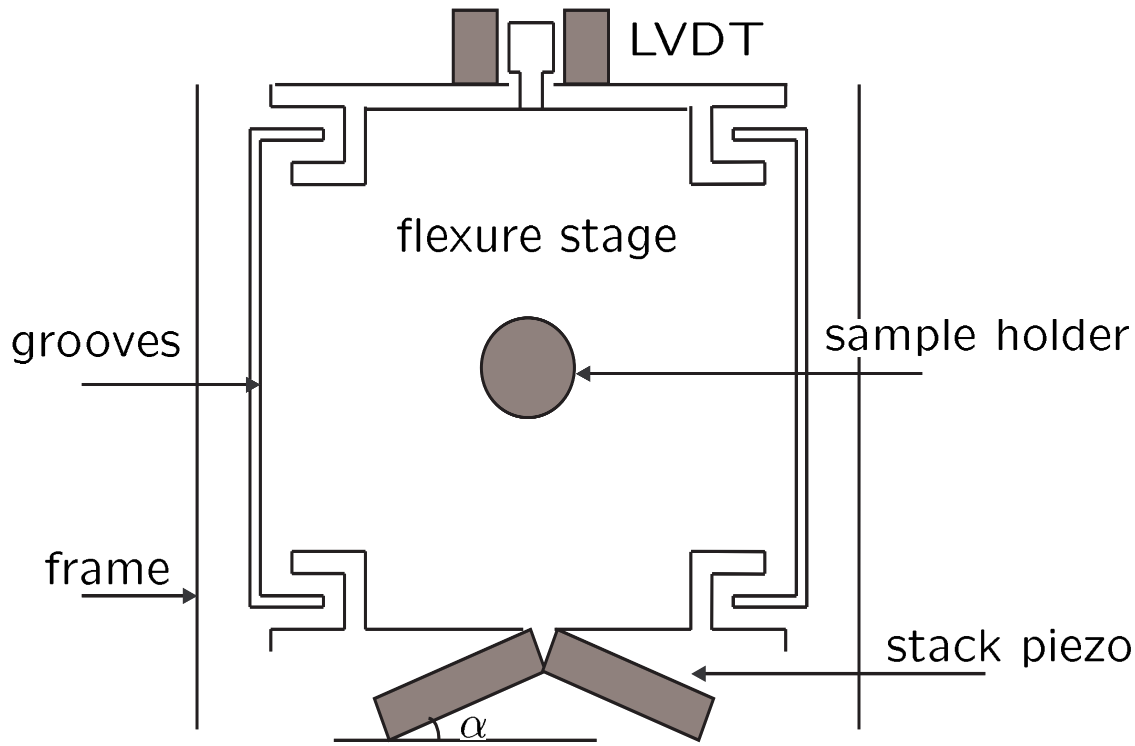


(a) (b)

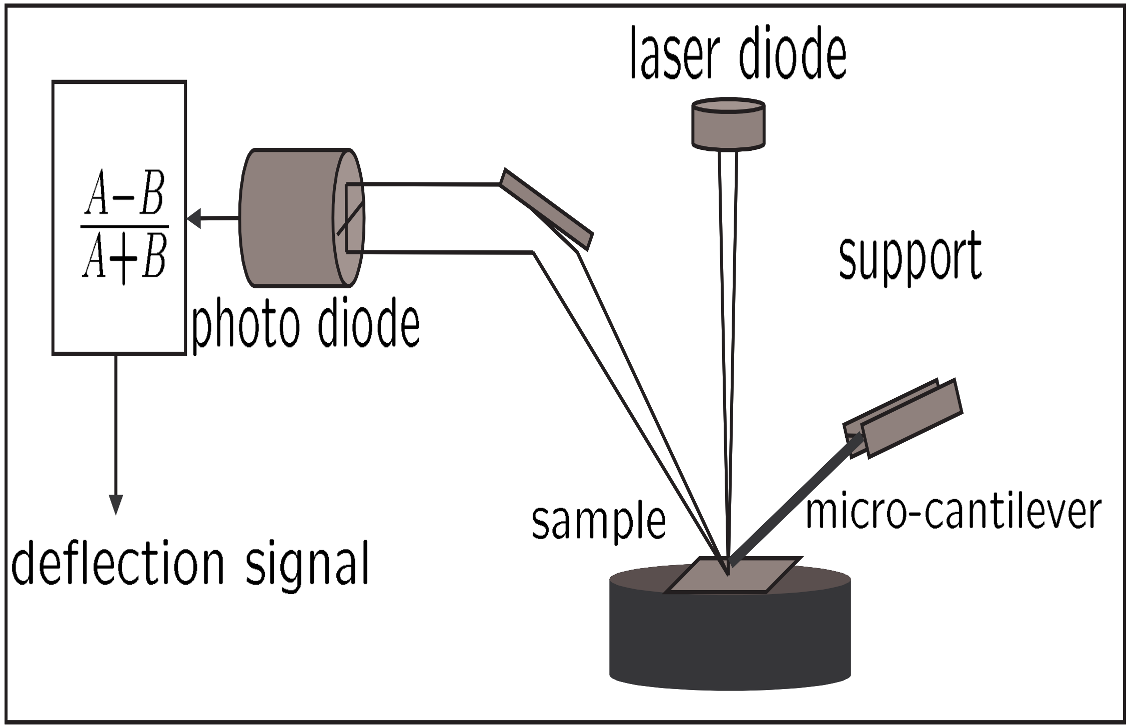
Serpentine Stage: Unassembled View



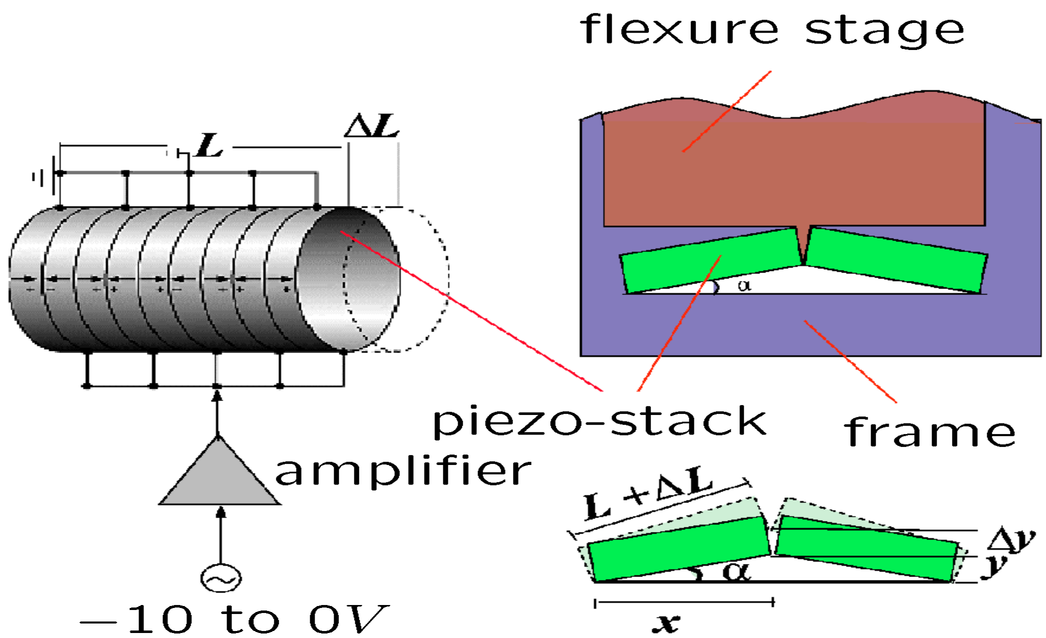
Serpentine Stage: Working Principle



Evaluation Stage

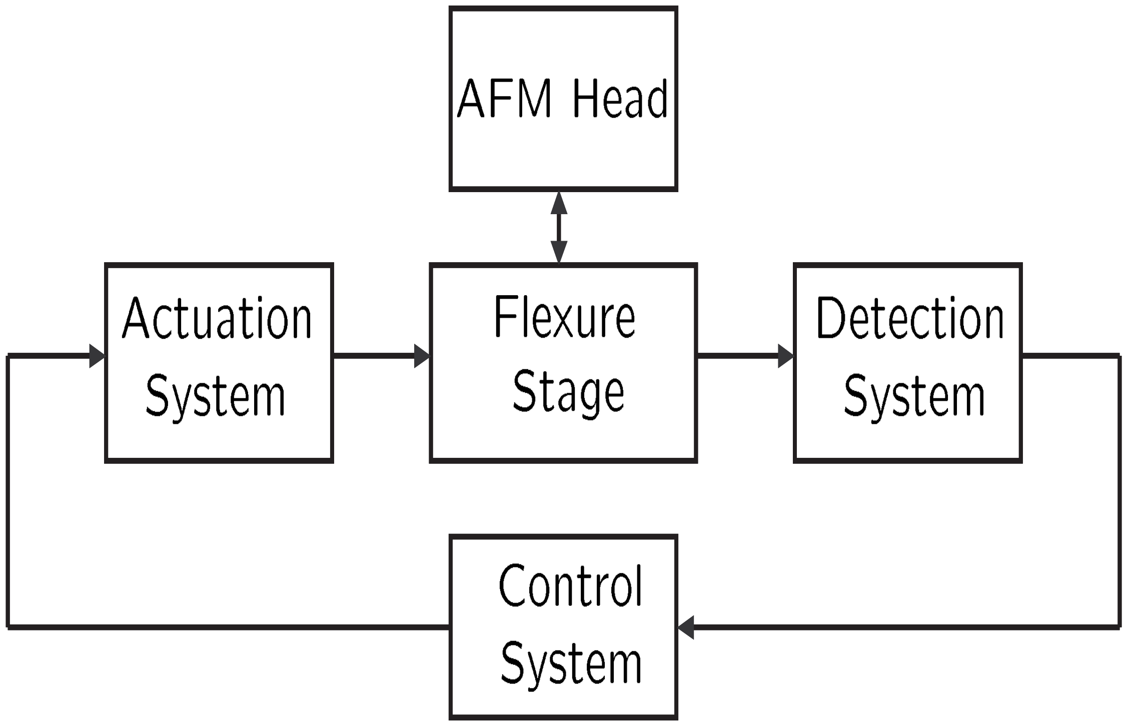


Piezo Actuators

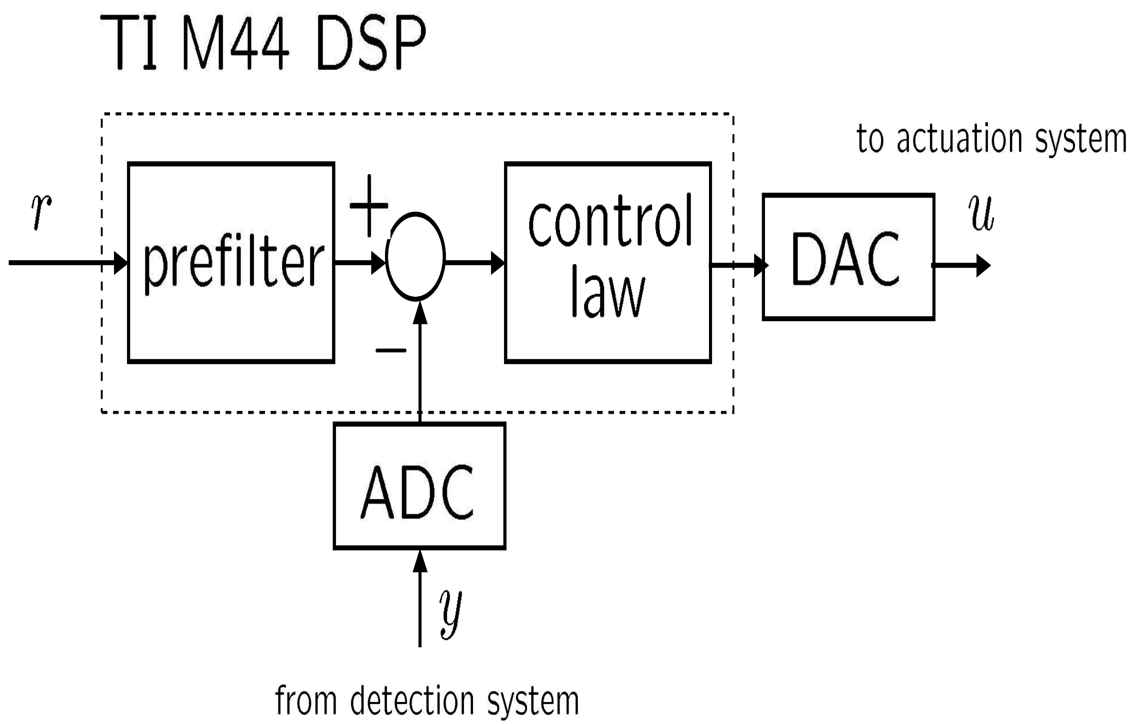


(a) (b)

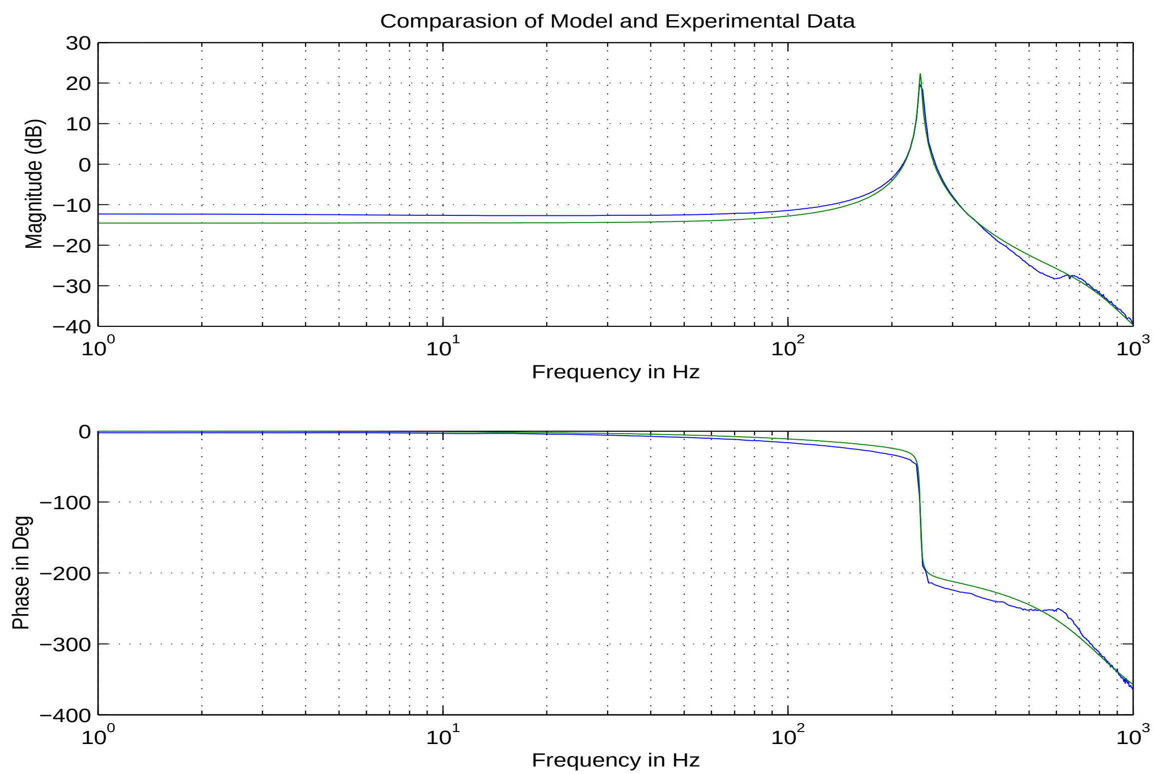
Block Diagram



Control Implementation



Obtaining a Model



- The frequency response of the plant with the input being the low voltage signal to the amplifier for the piezo actuators and the output being the LVDT sensor voltage was obatined. HP 3563 A control system analyzer was employed.

- This system analyzer stores a complex number corresponding to each frequency ω. Each complex number $H(j\omega)$ is the frequency response of the system at frequency ω.

- Matlab routine *invfreqs* can be used to fit a model to the frequency data.

Matlab Code

```
freq=load('freq.txt'); % defines the frequency vector

mag=load('mag1.txt'); %defines the magnitude in dB (corresponding to the
freq vector)

pha=load('phase1.txt');
```

freqr=freq*2*pi; mag=$10.^{(mag/20)};$

```
phar=unwrap(pha*pi/180);

H=mag.*exp(i*phar);

[num,den]=invfreqs(H,freqr,2,4);

Hfit=freqs(num,den,freqr);

magfit=abs(Hfit);
```

```
phafit=unwrap(angle(Hfit))*180/pi;
figure;
subplot(2,1,1)
hold on;
plot(freq,mag,freq,magfit);
title('Comparasion of Model and Experimental Data');
xlabel('Frequency in Hz');
ylabel('Magnitude');
subplot(2,1,2);
hold on;
plot(freq,phar*180/pi,freq,phafit);
```

```
xlabel('Frequency in Hz');

ylabel('Phase in Deg');

G=tf(num,den);
```

- The transfer function is given by

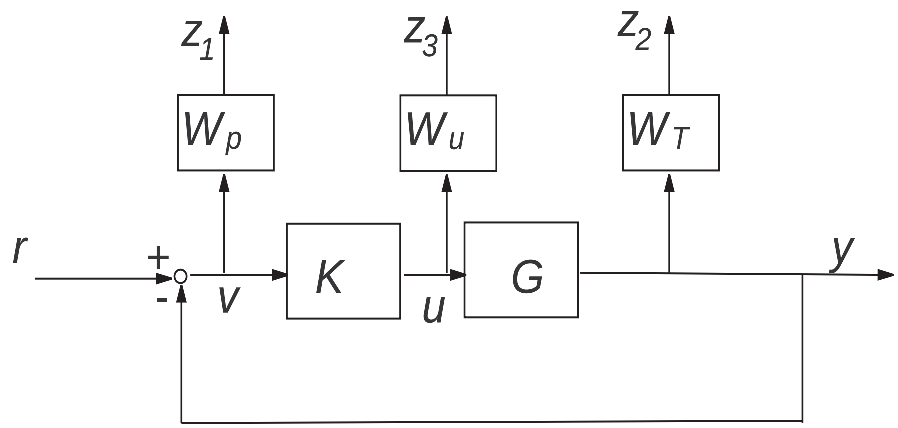
$$G(s) = \frac{97030.7242(s^2 - 1.44e004s + 1.06e008)}{(s^2 + 23.43s + 2.312e006)(s^2 + 3729s + 2.369e007)}$$

- The poles and zeros are at

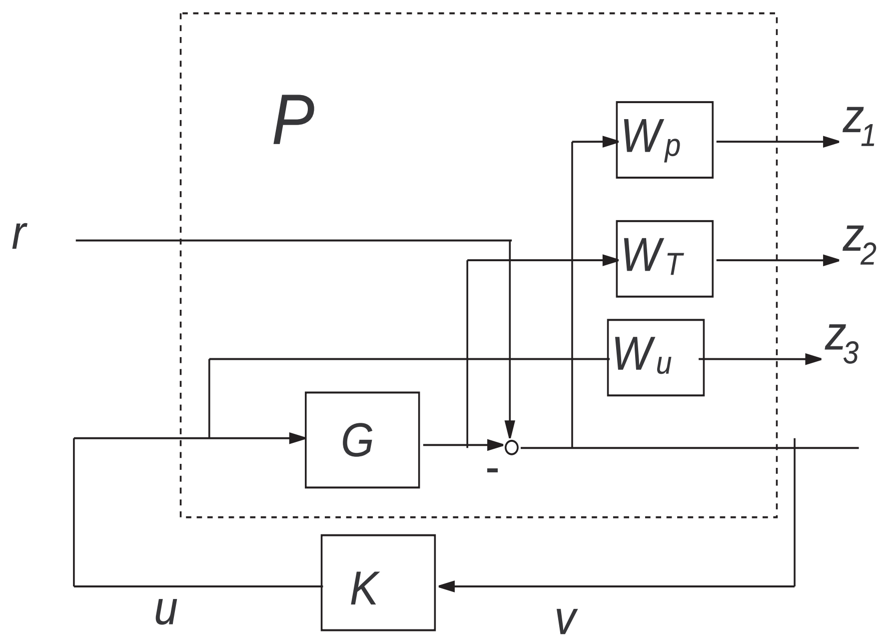
$$(1.0e+003)\begin{bmatrix} -1.8647 + 4.4958i, \\ -1.8647 - 4.4958i, \\ -0.0117 + 1.5206i, \\ -0.0117 - 1.5206i \end{bmatrix}, \quad (1.0e+003)\begin{bmatrix} 7.1993 + 7.3616i, \\ 7.1993 - 7.3616i \end{bmatrix}$$

- Presence of right half plane zeros.

Generalized Plant For The Stacked Problem



z_1
z_3
z_2
W_p
W_u
W_T
r
+
-
v
K
u
G
y



P
W_p
z_1
r
W_T
z_2
W_u
z_3
G
-
K
u
v

- The transfer function between r and z_1 is $W_P S$.

- The transfer function between r and z_2 is $W_T T$.

- The transfer function between r and z_3 is $W_u K S$.

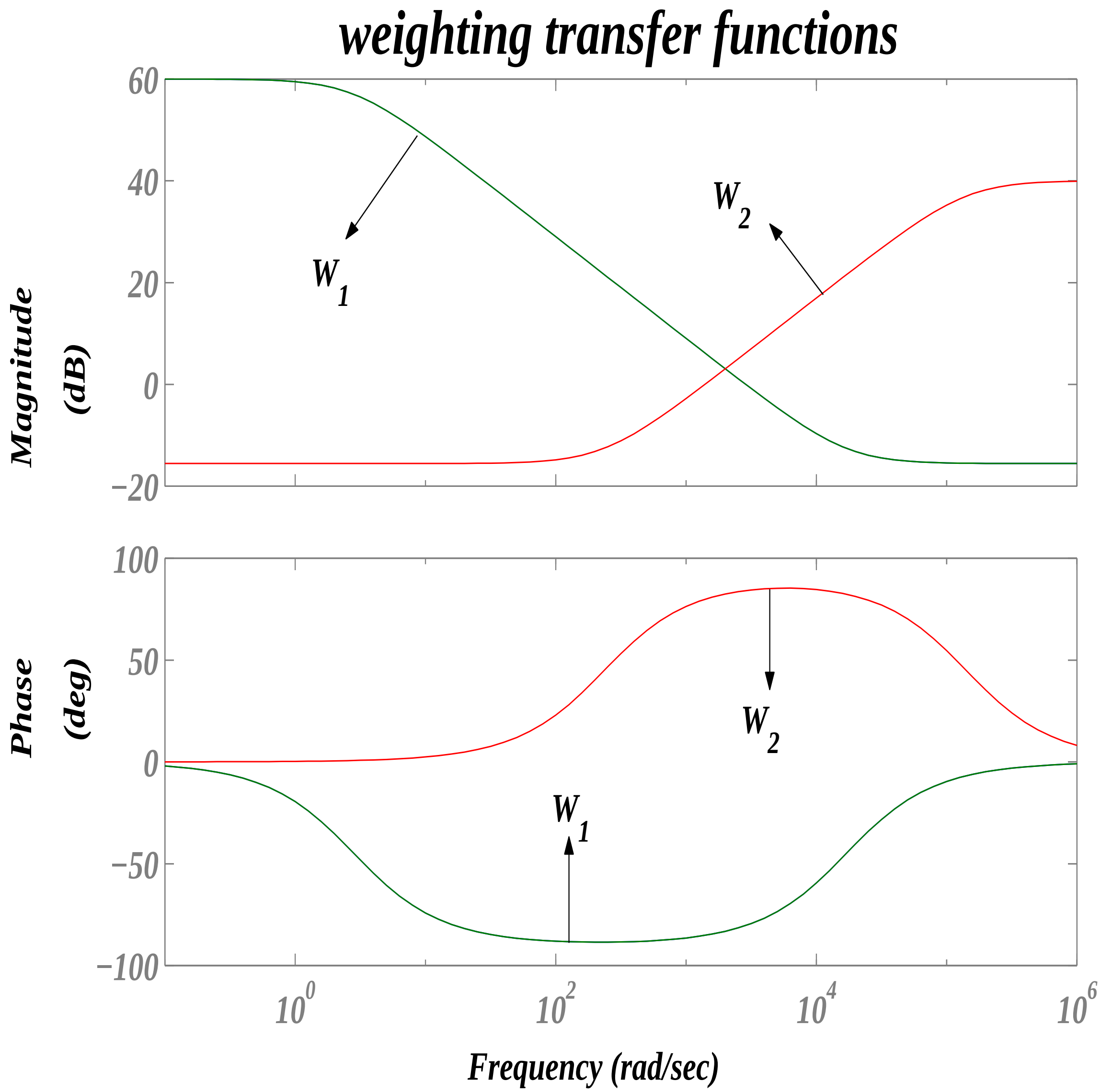
The regulated outputs are given by

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} W_p(r - Gu) \\ W_T G u \\ W_u u \end{bmatrix},$$

and the generalized plant, P is described by

$$\begin{bmatrix} z \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \begin{bmatrix} W_p & -W_pG \\ 0 & W_TG \\ 0 & W_u \end{bmatrix} \\ \begin{bmatrix} I & -G \end{bmatrix} \end{bmatrix}}_{=P} \begin{bmatrix} r \\ u \end{bmatrix}.$$

Weight Selection


weighting transfer functions
60
40
20
0
−20
Magnitude
(dB)
W
1
W
2
100
50
0
−50
−100
Phase
(deg)
W
2
W
1
10
0
10
2
10
4
10
6
Frequency (rad/sec)

- The transfer function, W_p, is chosen such that it has high gains at low frequencies and low gains at high frequencies. This scaling ensures that the sensitivity function is small at low frequencies, thus guaranteeing good tracking at the concerned frequencies. W_p was chosen to be a first order transfer function,

$$W_p = W_1(s) = \frac{0.1667s + 2827}{s + 2.827}.$$

- This transfer function is designed so that its inverse (an upper bound on the sensitivity function) has a gain of 0.1% at low frequencies ($< 1\ Hz$) and a gain of $\approx 5\%$ around $200\ Hz$.

- The weighting function W_p puts a lower bound on the bandwidth of the closed loop system but does not allow us to specify the roll off of the open loop system to prevent high frequency noise amplification and to limit the bandwidth to be below Nyquist frequency.

- Piezoactuators do not have any backlash or friction and therefore have very fine resolution. The resolution of the device, therefore, depends on the experimental environment and it is limited by thermal and electronic noise.

- In any closed loop framework the high resolution of the piezoactuators may be compromised due the introduction of the sensor noise (in this case the LVDT) into the system. Clearly this effect is absent in the open loop case.

- In the H_∞ paradigm these concerns of sensor noise rejection are reflected by introducing a weighted measure of the complementary sensitivity function, T, (which is the transfer function between the noise and the position y).

- In this case this weight was chosen to be

$$W_T = W_2 = \frac{s + 235.6}{0.01s + 1414}$$

which has high gains at high frequencies (note that noise is in the high frequency region).

- There is another interesting interpretation of this weighting function. It decides the resolution of the device. Resolution is defined as the variance of the output signal y, when the device is solely driven by the noise n; i.e., resolution is equal to the variance of Tn.

- W_T that guarantees lower roll off frequencies gives finer resolution. In this way, the trade-off between conflicting design requirements of high bandwidth tracking (characterized by low S, $T \approx 1$) and fine resolutions (characterized by low T) are translated to the design of weighting transfer functions W_p and W_T.

- The transfer function, KS was scaled by a constant weighting $W_u = 0.1$, to restrict the magnitude of the input signals such that they are within the saturation limits. This weighting constant gives control signals that are at most six times the reference signals.

Matlab Code *Defining the weights*

```
wbp=2*pi*450;
Mp=6;mp=1e-3;
mth=1e-2;mtl=1/6;wbt=0.5*wbp;
Mu=1e4;
muv=1/10; red=200;
nump=[1/Mp wbp];denp=[1 wbp*mp];
numt=[1 mtl*wbt];dent=[mth*1 wbt];
numu=[0 muv];denu=[0 1];
sysWp=tf(nump,denp);
sysWt=tf(numt,dent);
sysWu=tf(numu,denu);
```

```
P=[sysWp -sysWp*G;0 sysWt*G;0 sysWu;1 -G];
ssP=minreal(ss(P));
[aP,bP,cP,dP]=ssdata(ssP);
pckP=pck(aP,bP,cP,dP);
qt=1;gmin=0.1;gmax=15;tol=1e-3;epr=1e-12;epp=1e-8;rm=2;
nc=1;nm=1;
[K,cl,gf,ax,ay,hx,hy]=hinfsyn(pckP,nm,nc,gmin,gmax,tol,rm,epr,epp,qt);
[aK,bK,cK,dK]=unpck(K);
ssK=ss(aK,bK,cK,dK);
tfK=tf(ssK);
zpkK=zpk(tfK);
```

- The optimal γ value returned is 2.416. Note that this implies that

$$\left\| \begin{array}{c} W_pS \\ W_TT \\ W_uKS \end{array} \right\|_{\mathcal{H}_\infty} \le 2.416.$$

Thus it is not guaranteed that
$\|W_pS\|_{\mathcal{H}_\infty} \le 1,\ \|W_TT\|_{\mathcal{H}_\infty} \le 1$ and $\|W_u u\|_{\mathcal{H}_\infty} \le 1$.

- The controller transfer function is given by

$$277030168.45\frac{(s+1.414e5)(s^2+23.43s+2.31e6)(s^2+3729s+2.37e7)}{(s+1.15e7)(s+1.414e5)(s+5643)(s+2.827)(s^2+3135s+3.66e7)}.$$

Another Matlab routine is based on the *hinfopt* and *hinf* routines. The code to use these functions is

% Assuming that the weights (sysWp, sysWt, sysWu) and the plant transfer function G are defined as tf objects

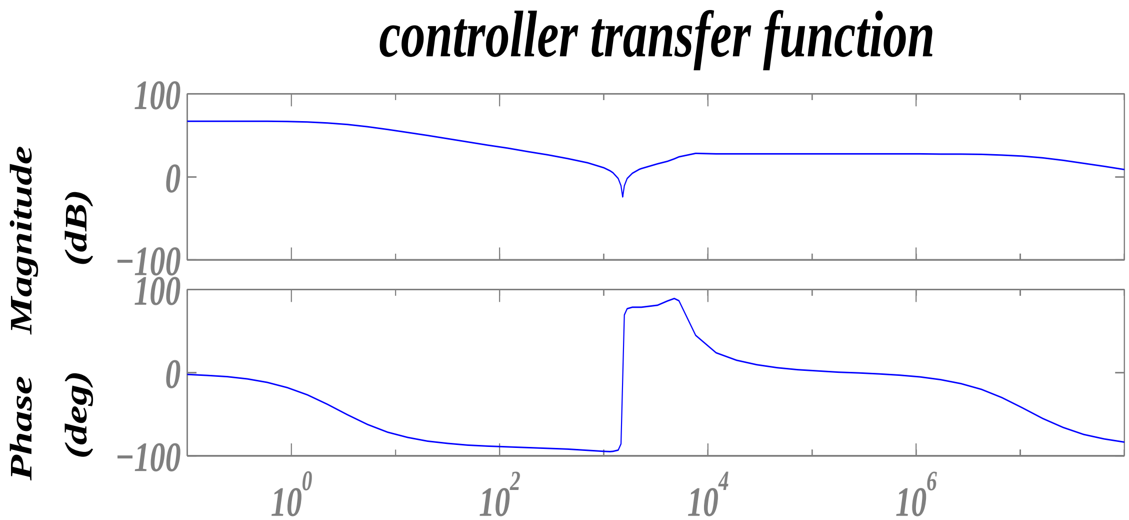
```
ssG=ss(G);

TSS=augtf(ssG,sysWp,sysWu,sysWt);

[gammaopt,ssf,sscl]=hinfopt(TSS);
```

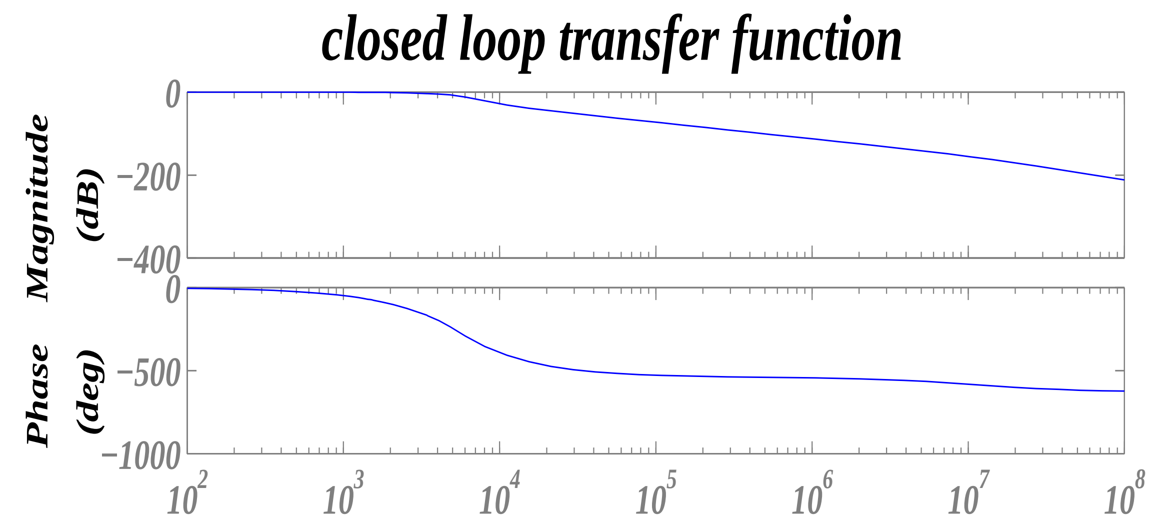
% gammopt is the optimal gamma value, ssf is the optimal controller (an ss object) and sscl is the optimal closed loop map (again an ss object)

- This yields an optimal γ value of $1/0.4102 = 2.438$. Note that the *gammaopt* value returned by the *hinfopt* command is the reciprocal of the *gamma* value returned by the *hinfsyn* command. Also, the optimal controllers as provided by the *hinfsyn* and *hinfopt* commands are not the same. The $\mathcal{H}_\infty$ optimal controllers are not unique.

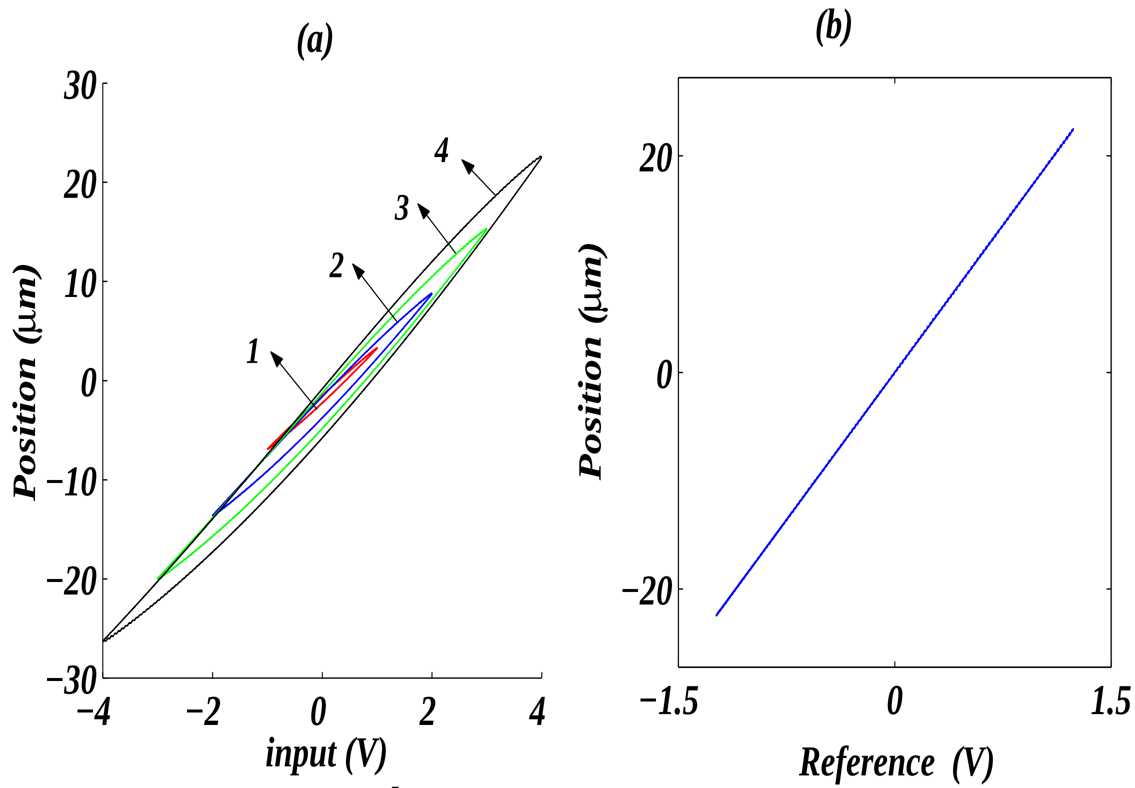
- The results provided are for the controller associated with the μ control tool box (the function *hinfsyn*).

Controller and Closed loop Transfer Functions


controller transfer function
Magnitude (dB)
100
0
−100
Phase (deg)
100
0
−100
10^0
10^2
10^4
10^6


closed loop transfer function
Magnitude (dB)
0
−200
−400
Phase (deg)
0
−500
−1000
10^2
10^3
10^4
10^5
10^6
10^7
10^8

Hysteresis



open loop

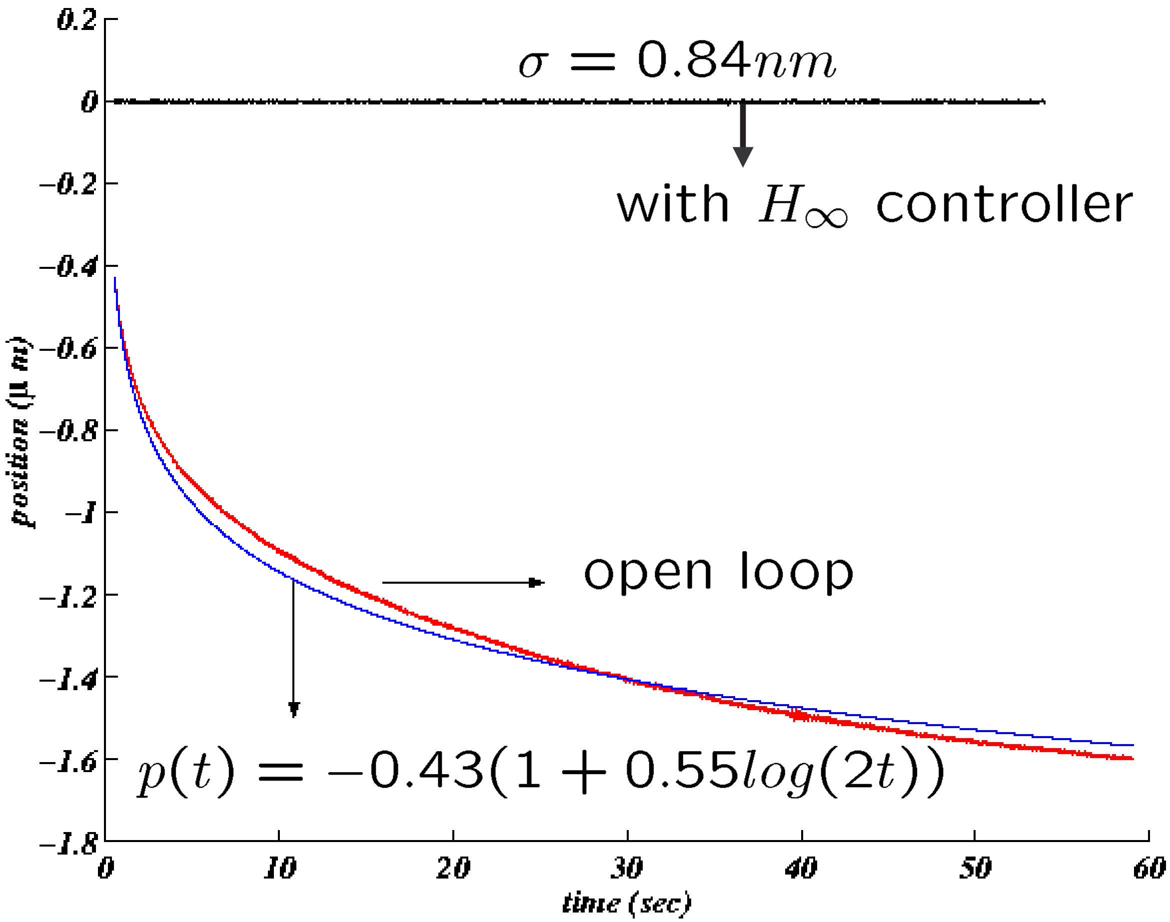
	max. out. hyst.	max. inp. hyst.
1	0.74 μm (7.2%)	0.14 V (5.8%)
2	2.09 μm (9.3%)	0.36 V (7.5%)
3	3.46 μm (9.8%)	0.56 V (7.7 %)
4	4.93 μm (10.0 %)	0.73 V (7.6 %)

closed loop

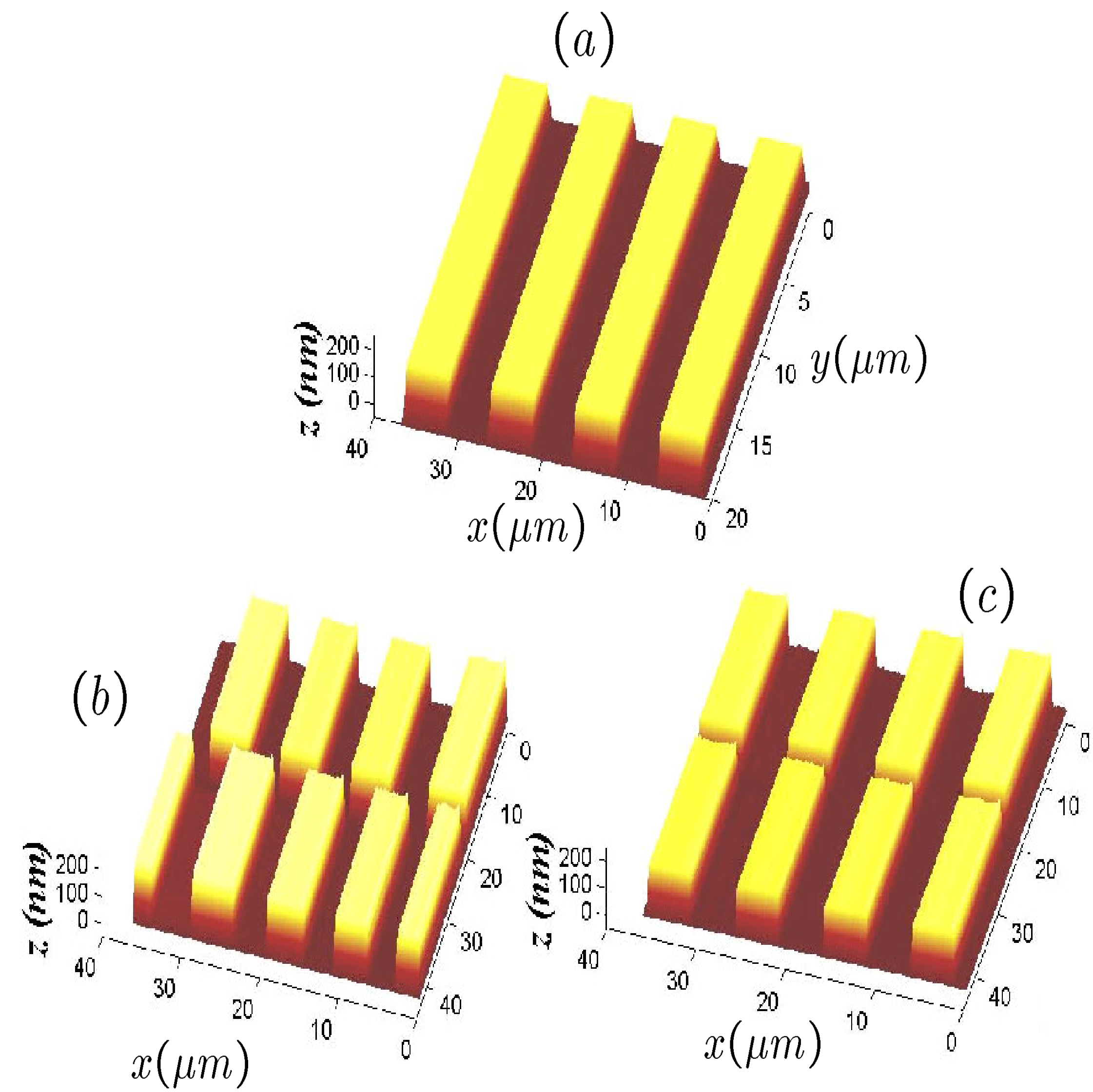
max. out. hyst.:
62.3 nm (0.14 %)

max. inp. hyst.:
2 mV (0.07 %)

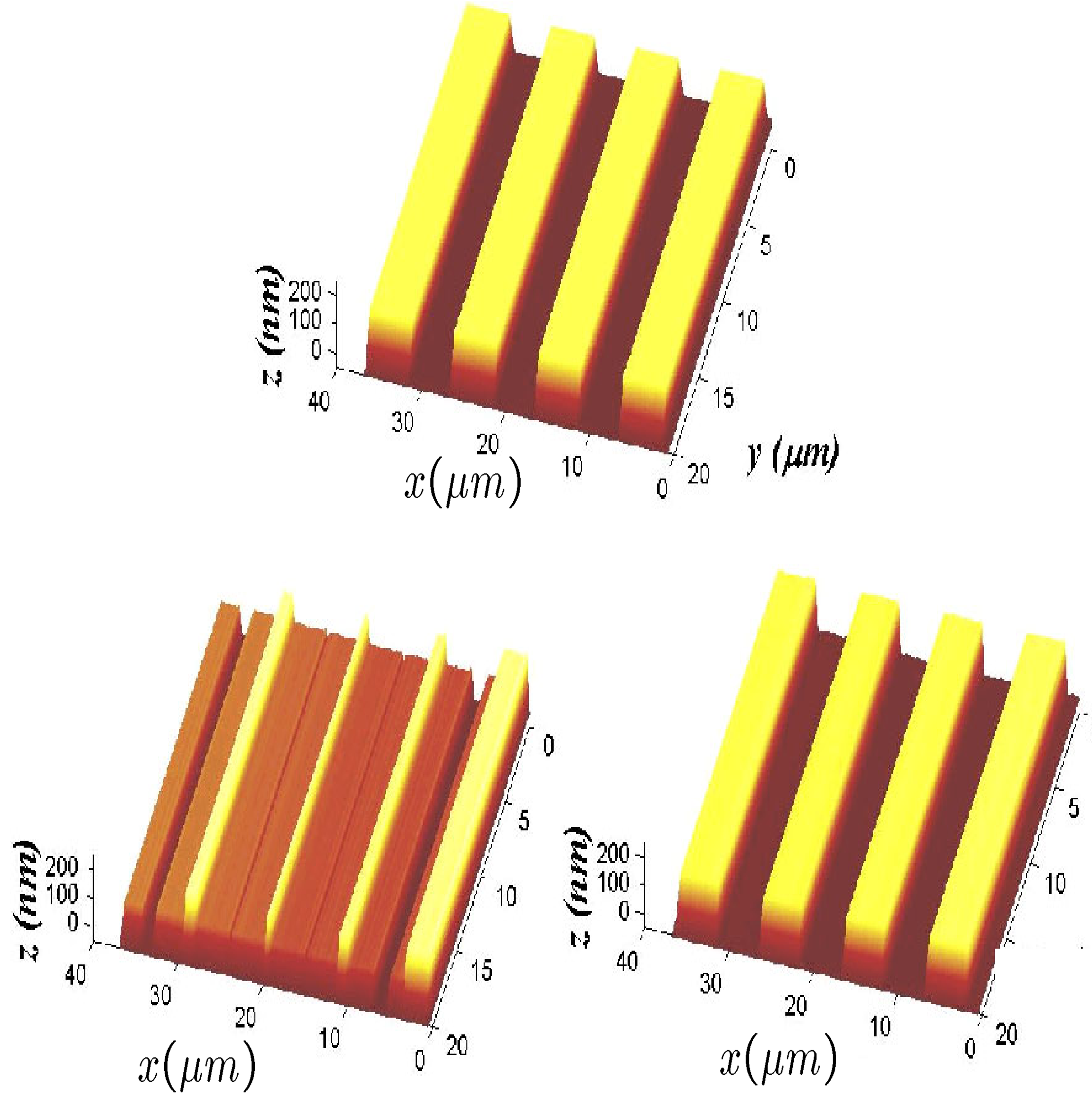
Creep


$\sigma = 0.84nm$
with H_∞ controller
open loop
$p(t) = -0.43(1 + 0.55log(2t))$
position (μ m)
time (sec)
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
0
10
20
30
40
50
60

Imaging: Closed and Open Loop


(a)
z (nm)
200
100
0
40
30
20
10
0
$x(\mu m)$
0
5
10
15
20
$y(\mu m)$
(b)
z (nm)
200
100
0
40
30
20
10
0
$x(\mu m)$
0
10
20
30
40
(c)
z (nm)
200
100
0
40
30
20
10
0
$x(\mu m)$
0
10
20
30
40

Imaging: Closed and open loop


z (nm)
200
100
0
40
30
20
10
0
$x(\mu m)$
0
5
10
15
20
y (μm)
z (nm)
200
100
0
40
30
20
10
0
$x(\mu m)$
0
5
10
15
20
z (nm)
200
100
0
40
30
20
10
0
$x(\mu m)$
0
5
10
20

Reading Assignment

Read the paper

S. Salapaka, A. Sebastian, J. P. Cleveland and M. V. Salapaka, ”High Bandwidth Nano-positioner: A Robust Control Approach”, Review of Scientific Instruments, Vol. 73, no. 9, pp. 3232-3241.

Fundamental Limitations For Single-input Single-output Systems

Definition 5. *(Analytic functions, holomorphic functions) Let Γ be a domain in C and let f be a function defined on Γ. Then f is said to be analytic or holomorphic at s_0 in C if $\frac{df}{ds}(s_0)$ exists. It is analytic or holomorphic in Γ if it analytic or holomorphic at all elements of Γ.*

Definition 6. *(Entire functions) A function is said to be entire if it is analytic on C.*

Example 1. *Rational functions on the complex plane are analytic everywhere on the the complex plane except at the poles.*

Definition 7. *(Rectifiable curve, simple curve, closed curve) A set Γ in the complex plane C is a rectifiable curve if there exists a continuously differentiable function $\gamma : [a, b] \subset R \to C$ such that $\Gamma = \gamma([a, b])$.*

A rectifiable curve Γ is a simple curve if it does not intersect itself. That is the associated function γ is such that $\gamma(x) \neq \gamma(y)$ if $x \neq y$ for all $x, y \in (a, b)$.

A rectifiable curve is closed if $\gamma(a) = \gamma(b)$.

Definition 8. *(Contour) A contour Γ is a collection of rectifiable curves Γ_j such that the final point of Γ_j is the initial point of Γ_{j+1}. Closed and simple contours are analogously defined as the corresponding definitions for curves.*

Definition 9. *(Integral) For a function f that is continuous on the domain S the integral along a rectifiable curve $\Gamma \subset S$ is defined as*

$$\int_\Gamma f(s)ds := \int_a^b f(\gamma(x))\frac{d\gamma}{dx}(x)dx,$$

where $\gamma([a,b]) = \Gamma$.

The integral over a contour is defined as

$$\int_\Gamma f(s)ds := \sum_{j=1}^{n} \int_{a_j}^{b_j} f(\gamma_j(x))\frac{d\gamma_j}{dx}(x)dx$$

where $\Gamma_j = \gamma_j([a_j, b_j]),\ j = 1, \ldots n$ *forms the contour* Γ.

Definition 10. *(Positively oriented contour)*

Consider a simple, closed contour formed by rectifiable curves $\Gamma_j = \gamma_j([a_j, b_j]),\ j = 1, \ldots n$.

Let x_0 *be such that* $x_0 \in [a_j, b_j]$ *such that* $\frac{d\gamma_j}{dx}(x_0) \neq 0$

If the vector obtained by rotating the tangent vector at x_0 *given by* $\frac{d\gamma_j}{dx}(x_0)$ *by* 90 *degrees anticlockwise points to the inside of the contour then the closed simple contour is positively oriented.*

Maximum Modulus Theorem

Theorem 6. *(Maximum Modulus theorem) Suppose that Ω is a non-empty, open, connected set in the complex plane and F is a function that is analytic in Ω. Suppose that F is not equal to a constant. Then $|F|$ does not attain its maximum value at an interior point of Ω*

A simple application of the above theorem is the following fact for a stable transfer function F:

$$\|F\|_{\mathcal{H}_\infty} = \sup_{\omega \in R} |F(j\omega)| = \sup_{Re(s) \geq 0} |F(s)|.$$

Cauchy's Theorem

Theorem 7. *(Cauchy's theorem) Consider the simply connected domain S that contains the simple, closed contour Γ that is positively oriented. If f is analytic in S then*

$$\int_{\Gamma} f(s)ds = 0.$$

Also, for any point $s_0 \in S$

$$\frac{1}{2\pi j}\int_{\Gamma} \frac{f(s)}{(s - s_0)}ds = f(s_0).$$

Weighted Cauchy's Theorem

Theorem 8. *Let F be analytic and of bounded magnitude on $\{s \in C | Re(s) \geq 0\}$. Let $s_0 = x + jy$ be a point such that $x > 0$. Then*

$$F(s_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(j\omega) \frac{x}{x^2 + (\omega - y)^2} d\omega.$$

Proof: Consider the Nyquist Contour $D(r)$ of radius r that includes s_0. From Cauchy's theorem we have that

$$F(s_0) = \frac{1}{2\pi j} \int_{D(r)} \frac{F(s)}{(s - s_0)} ds.$$

Note that $-\bar{s}_0 = -x + jy$ is in the strict left half plane and thus is not inside the Nyquist Contour. This implies that the function $\frac{F(s)}{s+\bar{s}_0}$ is analytic inside $D(r)$.

Thus using Cauchy's theorem it follows that

$$\frac{1}{2\pi j}\int_{D(r)}\frac{F(s)}{(s+\overline{s}_0)}ds = 0.$$

Subtracting the two integrals

$$
\begin{aligned}
F(s_0) &= \tfrac{1}{2\pi j}\int_{D(r)} F(s)\left(\tfrac{1}{(s-s_0)} - \tfrac{1}{(s+\bar{s}_0)}\right)ds \\
&= -\tfrac{1}{2\pi j}\int_{-r}^{r} F(j\omega)\left(\tfrac{1}{(j\omega-s_0)} - \tfrac{1}{(j\omega+\bar{s}_0)}\right)jd\omega + \\
&\quad \tfrac{1}{2\pi j}\int_{-\pi/2}^{\pi/2} F(re^{j\theta})\left(\tfrac{1}{(re^{j\theta}-s_0)} - \tfrac{1}{(re^{j\theta}+\bar{s}_0)}\right)rje^{j\theta}d\theta \\
&= -\tfrac{1}{2\pi j}\int_{-r}^{r} F(j\omega)\left(\tfrac{2x}{(j\omega-s_0)(j\omega+\bar{s}_0)}\right)jd\omega + \\
&\quad \tfrac{1}{2\pi j}\int_{-\pi/2}^{\pi/2} F(re^{j\theta})\left(\tfrac{2x}{(re^{j\theta}-s_0)(re^{j\theta}+\bar{s}_0)}\right)rje^{j\theta}d\theta \\
&= \tfrac{1}{\pi}\int_{-r}^{r} F(j\omega)\tfrac{x}{x^2+(\omega-y)^2}d\omega + \\
&\quad \tfrac{1}{\pi}\int_{-\pi/2}^{\pi/2} F(re^{j\theta})\left(\tfrac{x}{(re^{j\theta}-s_0)(re^{j\theta}+\bar{s}_0)}\right)re^{j\theta}d\theta \\
&=: I_1(r) + I_2(r)
\end{aligned}
$$

Note that as $r \to \infty$, $I_1(r) \to \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{x}{x^2+(\omega-y)^2}d\omega.$

Note that

$$\begin{aligned} |I_2(r)| &\leq \tfrac{1}{\pi}\int_{-\pi/2}^{\pi/2}|F(re^{j\theta})|(\tfrac{x}{|(e^{j\theta}-r^{-1}s_0)|\ |(e^{j\theta}+r^{-1}\overline{s}_0)|})r^{-1}d\theta \\ &\leq \tfrac{1}{r}\|F\|_{\mathcal{H}_\infty}\int_{-\pi/2}^{\pi/2}(\tfrac{x}{|(e^{j\theta}-r^{-1}s_0)|\ |(e^{j\theta}+r^{-1}\overline{s}_0)|})d\theta \\ &\leq Const\times\tfrac{1}{r} \end{aligned}$$

Thus $I_2(r)\to 0$ as $r\to\infty$.

Thus

$$F(s_0)=\lim_{r\to\infty}I_1(r)=\frac{1}{\pi}\int_{-\infty}^{\infty}F(j\omega)\frac{x}{x^2+(\omega-y)^2}d\omega.$$

This proves the lemma. ■

All-pass and Minimum Phase Transfer Function

Definition 11. *(All pass transfer function) A stable proper rational function G is all pass if*

$$|G(j\omega)| = 1, \forall \omega \in R.$$

It can be shown that if G is an all pass transfer function then s_0 is a pole of G if and only if $-\overline{s}_0$ is a zero. Thus all pass functions have the form

$$G(s) = \Pi_{i=1}^{n} \frac{s + \overline{s}_n}{s - s_n}.$$

Definition 12. *(Minimum-phase transfer functions) A proper rational function is minimum phase if all its zeros are in the strict left half plane.*

All-pass/Minimum Phase Factorization

Theorem 9. *(All-pass/minimum phase factorization) Every stable proper rational function G admits a factorization of the form*

$$G = G_{ap} G_{mp}$$

where G_{ap} is all pass and G_{mp} is minimum phase.

Proof: Let $G(s) = K\frac{(s-z_1)\ldots(s-z_n)}{(s-p_1)\ldots(s-p_k)}$. As G is stable it is clear that p_i are all in the left half plane. Without loss of generality assume that $z_1, z_2, \ldots, z_m$ are the zeros in the right half plane (we will assume that there are no zeros on the $j\omega$ axis). Then it is clear that

$$G(s) = \underbrace{[\Pi_{i=1}^{m} \frac{s - z_i}{s + \overline{z}_i}]}_{G_{ap}} \underbrace{[K\frac{\Pi_{i=1}^{m}(s + \overline{z}_i)\Pi_{i=m+1}^{n}(s - z_i)}{\Pi_{i=1}^{k}(s - p_i)}]}_{G_{mp}},$$

where as z_i is in the strict right half plane, $-\overline{z}_i$ is in the strict left half plane.

Clearly G_{ap} is all pass and G_{mp} is minimum phase. $\blacksquare$

A Lemma

Lemma 1. *Let $G(s)$ be a stable proper transfer function with the factorization $G = G_{ap}G_{mp}$ with G_{ap} being all-pass and G_{mp} being minimum phase. Let $s_0 = x + jy$ be in the strict right half plane. Then*

$$\log|G_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|G(j\omega)| \frac{x}{x^2 + (\omega - y)^2} d\omega.$$

Proof: Let $F := \log(G_{mp})$. As G_{mp} is analytic in the right half plane and has no zeros in the right half plane it follows that F is analytic in the right half plane. Applying Lemma 8 it follows that

$$F(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega) \frac{x}{x^2 + (\omega - y)^2} d\omega.$$

Taking the real part on both sides we have

$$Re(F(s_0)) = \frac{1}{\pi}\int_{-\infty}^{\infty} Re(F(j\omega))\frac{x}{x^2+(\omega-y)^2}d\omega. \tag{1}$$

Note that $G_{mp} = e^F = e^{Re(F)}e^{jImg(F)}$. Thus $|G_{mp}| = e^{Re(F)}$ and $\log|G_{mp}| = Re(F)$.

It follows from (1) that

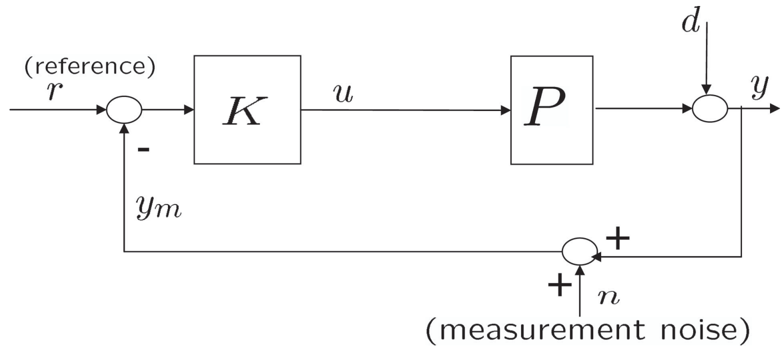
$$\log|G_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|G_{mp}(j\omega)|\frac{x}{x^2+(\omega-y)^2}d\omega.$$

Noting that $|G_{mp}(j\omega)| = |G(j\omega)|$ it follows that

$$\log|G_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|G(j\omega)|\frac{x}{x^2+(\omega-y)^2}d\omega.$$

$\blacksquare$

Fundamental Limitations on Performance



Let $L := GK$.

- We have seen that typical performance requirements need $S = \frac{1}{1+L}$ to be small for good tracking and disturbance rejection.

- It is desired that $T = I - S$ be small for good noise rejection.

Given a certain set of objectives it is desirable to evaluate the feasibility of the specifications that are targeted.

Cautionary Example

The importance of fundamental limitations is highlighted by the following example that concerns the design of X-29 aircraft. Considerable design effort was directed towards designing a controller that provides a phase margin of at least 45 degrees. However, a simple argument based on a result to be developed that utilizes the presence of an unstable pole and a right half plane zero would have indicated the infeasibility of such a requirement. Clearly utilization of results that yield such an analysis can lead to significant economy in time, effort and cost.

Waterbed Effect I

Theorem 10. *(Waterbed effect I) Let L have relative degree two and let L have N_p poles in the right half given by $p_1, \ldots, p_{N_p}$. If the closed-loop system is stable then $S = \frac{1}{1+L}$ satisfies*

$$\int_0^\infty \ln |S(j\omega)| d\omega = \pi \sum_{i=1}^{N_p} Re(p_i).$$

Proof: Note that the poles of L are the zeros of S. Thus p_i are the right half plane zeros of S. Thus

$$S_{ap}(s) = \Pi_{i=1}^{N_p} \frac{s - p_i}{s + \overline{p}_i}.$$

From Lemma 1 it follows for any $x > 0$ that

$$\ln|S_{mp}(x)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \ln|S(j\omega)|\frac{x}{x^2+\omega^2}d\omega = \frac{1}{\pi}\int_{0}^{\infty} \ln|S(j\omega)|\frac{2x}{x^2+\omega^2}d\omega.$$

Thus it follows that

$$\int_{0}^{\infty} \ln|S(j\omega)|\frac{x^2}{x^2+\omega^2}d\omega = \frac{\pi}{2}x\ln|S_{mp}(x)|.$$

Therefore

$$\lim_{x\to\infty}\int_{0}^{\infty} \ln|S(j\omega)|\frac{x^2}{x^2+\omega^2}d\omega = \lim_{x\to\infty}\frac{\pi}{2}x\ln|S_{mp}(x)|$$

which implies that

$$\begin{aligned}
\int_0^\infty \ln|S(j\omega)|d\omega &= \lim_{x\to\infty} \tfrac{\pi}{2} x \ln|S_{mp}(x)| \\
&= \tfrac{\pi}{2}(\lim_{x\to\infty} x \ln|S(x)| - \lim_{x\to\infty} x \ln|S_{ap}(x)|) \\
&= \tfrac{\pi}{2}(0 + \textstyle\sum_{i=1}^{N_p} Re(p_i)),
\end{aligned}$$

where $\lim_{x\to\infty} x\ln|S(x)|) = 0$ follows from the hypothesis that L has relative degree atleast two.

■

Waterbed Effect II

Theorem 11. *(Waterbed effect II; Weighted Sensitivity Integral) Let L have N_p poles in the right half given by $p_1, \ldots, p_{N_p}$. Let $z = x + jy$ be any zero of L in the strict right half plane (that is $x > 0$). If the unity feedback system is stable then $S = \frac{1}{1+L}$ is such that*

$$\int_0^\infty \ln|S(j\omega)|(\frac{x}{x^2 + (\omega - y)^2)} + (\frac{x}{x^2 + (\omega + y)^2})d\omega = \pi \ln(\Pi_{i=1}^{N_p} \left|\frac{z + \overline{p}_i}{z - p_i}\right|).$$

Proof:

Note that the poles of L are the zeros of S. Thus p_i are the right half plane zeros of S. Thus

$$S_{ap}(s) = \Pi_{i=1}^{N_p} \frac{s - p_i}{s + \overline{p}_i}.$$

Using Lemma 1 it follows that

$$\begin{aligned}\ln|S_{mp}(z)| &= \tfrac{1}{\pi}\int_{-\infty}^{\infty}\ln|S(j\omega)|\tfrac{x}{x^2+(\omega-y)^2}d\omega.\\ &= \tfrac{1}{\pi}\int_{0}^{\infty}\ln|S(j\omega)|(\tfrac{x}{x^2+(\omega-y)^2}+\tfrac{x}{x^2+(\omega+y)^2})d\omega.\end{aligned}$$

This implies that

$$\begin{aligned}\tfrac{1}{\pi}\int_{0}^{\infty}\ln|S(j\omega)|(\tfrac{x}{x^2+(\omega-y)^2}+\tfrac{x}{x^2+(\omega+y)^2})d\omega &= \ln|S_{mp}(z)| = \ln|\tfrac{S(z)}{S_{ap}(z)}|\\ &= \ln\tfrac{1}{|S_{ap}(z)|}\\ &= \ln(\Pi_{i=1}^{N_p}\left|\tfrac{z+\overline{p}_i}{z-p_i}\right|)\end{aligned}$$

This proves the theorem. ■

Bounds on Weighted S and T Transfer Functions

Theorem 12. *Supoose L has right half plane poles and zeros at $p_1, \ldots, p_{N_p}$ and $z_1, \ldots, z_{N_z}$ respectively. If the closed-loop system is stable then*

1. $\|w_p S\|_{\mathcal{H}_\infty} \geq \max_j \{ |w_p(z_j)| \; \Pi_{i=1}^{N_p} \left| \frac{z_j + \overline{p}_i}{z_j - p_i} \right| \}.$

2. $\|w_T T\|_{\mathcal{H}_\infty} \geq \max_i \{ |w_T(p_i)| \; \Pi_{j=1}^{N_z} \left| \frac{\bar{z}_j + p_i}{z_j - p_i} \right| \}.$

Proof:

Note that for any z_j

$$\begin{aligned} \|w_pS\|_{\mathcal{H}_\infty} = \sup_{\omega\in R}|w_p(j\omega)S(j\omega)| &= \sup_{\omega\in R}|w_p(j\omega)S_{mp}(j\omega)| \\ &= \sup_{Re(s)\geq 0}|w_p(s)S_{mp}(s)| \\ &\geq |w_p(z_j)S_{mp}(z_j)| \\ &= |w_p(z_j)\frac{S(z_j)}{S_{ap}(z_j)}| = |w_p(z_j)|\Pi_{i=1}^{N_p}\left|\frac{z_j+\overline{p}_i}{z_j-p_i}\right|. \end{aligned}$$

The third equality above from from the maximum-modulus theorem (Theorem 6) and the last equality follows by noting that as z_j is a zero of L, $S(z_j)=\frac{1}{1+L(z_j)}=1$. This proves the first part of the theorem. The proof for T is similar.

∎

Note that the terms $\left|\frac{z_j+\overline{p}_i}{z_j-p_i}\right| \geq 1$ for all relevant i and $j's$.

Bandwidth Limitations For Typical Weights

Note that for achieving the objectives of

- $\|S\|_{\mathcal{H}_\infty} \leq M$ and
- $|S(j\omega)| \leq m_p$ for all $\omega \leq \omega_B$

an appropriate weight is

$$w_p = \frac{s/M_p + \omega_B}{s + \omega_B m_p}.$$

The following corollary takes the limiting case of $m_p = 0$ and $M_p = 2$.

Corollary 1. *Let z be any right half plane zero of L. Let*

$$w_p = \frac{s/M_p + \omega_B}{s + \omega_B m_p}.$$

Then for the performance objective

$$\|w_p S\|_{\mathcal{H}_\infty} \leq 1$$

to be achieved the following conditions have to be satisfied

- *If z is real then*

  $$\omega_B < z\frac{1 - 1/M_p}{1 - m_p}.$$

  In particular if $M_p = 2$ and $m_p = 0$ then

  $$\omega_B < z/2.$$

- *If z is purely imaginary and $M_p = 2$ and $m_p = 0$ then*

  $$\omega_B < |z|\frac{\sqrt{3}}{2}.$$

Proof: From Theorem 12 we have that

$$\|w_p S\|_{\mathcal{H}_\infty} \geq \max_j \{|w_p(z_j)| \, \Pi_{i=1}^{N_p} \left|\frac{z_j + \overline{p}_i}{z_j - p_i}\right| \geq |w_p(z)| \Pi_{i=1}^{N_p} \left|\frac{z_j + \overline{p}_i}{z_j - p_i}\right| \geq |w_p(z)|.$$

Thus if the performance specification

$$\|w_p S\|_{\mathcal{H}_\infty} \leq 1,$$

has to be achieved then it is necessary that $|w_p(z)| < 1$. Thus $|z/M_p + \omega_B| < |z + \omega_B m_p|$ has to be satisfied. If z is real then this implies that

$$\omega_B < z\frac{1 - 1/M_p}{1 - m_p},$$

whereas if z is purely imaginary with $M = 2$ and $m_p = 0$ then

$$\omega_B < |z|\frac{\sqrt{3}}{2}.$$

This proves the corollary. ■

Note that the weight on T should ensure that T is small at high frequencies.

- $|T(j\omega)| < 1/M_T$ for all $\omega < \omega_T - \Delta\omega$
- $|T(j\omega)| < m_T$ for all $\omega > \omega_T + \Delta\omega$

where typically $1/M_T \approx 1$ and thus does not conflict with the sensitivity weighting, m_T is small forcing T to be small in the high frequency region. A typical weighting function has the form

$$w_T = \frac{s + (1/M_T)\omega_T}{m_T s + \omega_T}.$$

The specifications on T can be achieved by imposing

$$\|T\|_{\mathcal{H}_\infty} \leq \frac{1}{w_T(j\omega)}$$

which holds if and only if

$$\|w_T T\|_{\mathcal{H}_\infty} \leq 1$$

Corollary 2. *Let p be any right half plane pole of L. Let*

$$w_T = \frac{s + (1/M_T)\omega_T}{\omega_T}$$

where we have set $m_T = 0$. Then for the performance objective

$$\|w_T T\|_{\mathcal{H}_\infty}$$

to be achieved the following conditions have to be satisfied

- *If p is real then*

$$\omega_T > p\frac{M_T}{M_T - 1}.$$

In particular if $M_T = 2$ then

$$\omega_T > 2p$$

- *If p is purely imaginary then*

$$\omega_T > |p|\frac{M_T}{\sqrt{M_T^2 - 1}}.$$

In particular if $M_T = 2$ then

$$\omega_T > 1.15|p|$$

Proof: From Theorem 12 we have that

$$\|w_T T\|_{\mathcal{H}_\infty} \geq \max_i \{|w_T(p_i)| \, \Pi_{j=1}^{N_z} \left|\frac{z_j + \overline{p}_i}{z_j - p_i}\right| \geq |w_T(p)| \Pi_{j=1}^{N_z} \left|\frac{z_j + \overline{p}_i}{z_j - p_i}\right| \geq |w_T(p)|.$$

Thus if the performance specification

$$\|w_T T\|_{\mathcal{H}_\infty} \leq 1,$$

has to be achieved then it is necessary that $|w_T(p)| < 1$. Thus $|p + (1/M_T)\omega_T| < |\omega_T|$ has to be satisfied.

The rest of the proof follows from this condition. ■

Bandwidth Limitation: Crossover Frequency

- Let z be the zero of $L(s)$ in the right half plane closest to the $j\omega$ axis. Then from Corollary 1 it follows that the crossover frequency has to be chosen such that $\omega_c < \frac{z}{2}$.

- Let p be the pole of $L(s)$ in the right half plane farthest from the $j\omega$ axis. Then from Corollary 2 it follows that the crossover frequency has to be chosen such that $\omega_c > 2p$.

This would imply that ω_c has to satisfy

$$2p \leq \omega_c \leq \frac{z}{2}.$$

This would necessarily imply that $z > 4p$ to achieve good performance. In case this is not satisfied no controller will yield satisfactory performance.

Bandwidth Limitation Imposed by Disturbance Rejection

The error due to disturbance is given by

$$e = SG_d d.$$

If G_d is appropriately scaled then the objective of disturbance rejection is captured by

$$|e(\omega)| \leq 1, \text{ whenever } |d(\omega)| \leq 1.$$

In other words the objective is to ensure

$$\|SG_d\|_{\mathcal{H}_\infty} \leq 1.$$

A typical G_d has low frequency content. Let ω_d be the value such that $|G_d(\omega_d)| = 1$ that is ω_d is the frequency at which G_d crossed the $0dB$ line from above.

- From Theorem 12 it follows that

$$\|SG_d\|_{\mathcal{H}_\infty} \geq |G_d(z)|$$

where z is any right half plane zero of $L(s)$. Thus for good disturbance rejection it is needed that $|G_d(z)| < 1$.

- $\|SG_d\|_{\mathcal{H}_\infty} \leq 1$ implies that

$$|S(j\omega)| \leq \frac{1}{|G_d(\omega)|} \forall \omega.$$

Note that $|G_d(\omega))| > 1$ for all $\omega < \omega_d$. Thus it follows that $|S(j\omega)| < 1$ for all $\omega < \omega_d$. This would imply that $\omega_B > \omega_d$. Thus good disturbance rejection *requires* that the controller ensure that $\omega_B > \omega_d$.

Bandwidth Limitation Imposed by Input Bounds

Suppose we need the following condition to be satisfied:

$$|e(j\omega)| \leq 1 \text{ and } |u(j\omega)| \leq 1, \text{ whenever } |d(j\omega)| \leq 1 \text{ and } r = n = 0.$$

Note that

$$e = y - r = G_d d + Gu - r.$$

Assuming the needed condition (that is $|e(\omega)| < 1$ when $d(\omega) < 1$ and $r = n = 0$) is satisfied we have for any $|d(\omega)| < 1$ that

$$\begin{aligned} |Gu| = |e - G_d d| &\geq |G_d d| - |e| \\ &\geq |G_d|\ |d| - 1 \end{aligned}$$

Clearly the above condition holds for any d with $|d| = 1$ which implies that

$$|G| \geq |G|\ |u| \geq |G_d| - 1.$$

Bandwidth Limitation Imposed by Input Bounds for Unstable Plants

When the plant is unstable more restrictive conditions can be derived due to disturbance rejection. Note that the map between the control signal u and the disturbance d is given by

$$u = -KSG_d d = -G^{-1}TG_d d.$$

From Corollary 2 it follows that if p is a right half plane pole

$$\omega_T > p\frac{M_T}{M_T - 1} > p.$$

It follows that

$$|T| > 1, \ \forall \omega < p.$$

As $u = -TG^{-1}G_d d$ if the condition $|u| < 1$ whenever $|d| < 1$ needs to be satisfied then

$$|G| > |G_d|, \ \forall \omega < p.$$

Limitation Imposed by Reference Tracking

Assume that the references r to be tracked are well modeled as $r = R\tilde{r}$ where $|\tilde{r}(\omega)| \leq 1$. The performance objective is that

$$\text{if } |\tilde{r}(\omega)| < 1, \ \text{ for all } \omega \text{ then } |e(\omega)| < 1 \text{ and } |u(\omega)| < 1 \text{ for all } \omega < \omega_r.$$

If the above condition is satisfied and $|\tilde{r}| < 1$ then

$$|Gu| = |R\tilde{r} + e| \geq |R\tilde{r}| - |e| \geq |R\tilde{r}| - 1 \text{ for all } \omega < \omega_r.$$

The above relationship is also satisfied for any $\tilde{r}$ that is such that $|\tilde{r}(\omega)| \leq 1$, and $|G\tilde{r}| = |G|$. Thus we have

$$|G|\ |u| \geq |R| - 1 \text{ for all } \omega < \omega_r$$

As $|u| < 1$ it follows that

$$|G| \geq |R| - 1 \text{ for all } \omega < \omega_r.$$

Robust Stability for SISO systems

Introduction

- A control system is said to be robust if it is insensitive to the differences between the actual system and the model used to design the controller.

- The differences between the model and the actual plant is called the model uncertainty.

In the robust control paradigm the key concept is to design controllers that fulfill the specifications even for the *worst case* uncertainty. The approach that is pursued is

- Characterize the uncertainty mathematically.

- Analyze and synthesize controllers that achieve *Robust stability* (RS), that is analyze and synthesize controllers that ensure stability of the closed loop for all plants in the uncertainty set.

- Analyze and synthesize controllers that achieve *Robust performance* (RP), that is analyze and synthesize controllers that ensure stability and performance of the closed loop for all plants in the uncertainty set.

Sources of Uncertainty

- *Nonlinearities:* Note that a central design criteria for the robust control paradigm is the use of *linearity.* However, most plants exhibit nonlinear behavior. This leads to uncertainty.

- Uncertain parameters: Some of the parameters are uncertain in the model.

- *Measurement equipment:* Note that the measurement device has finite resolution and the equipment used to obtain the frequency response has limited capabilities. Thus often it is impossible to ascertain the model and high frequency where even the model order and structure cannot be determined.

- *Undermodeling:* Often the detailed and precise model of the plant is of very high order making it unsuitable for engineering purposes. Thus a lower order model is chosen resulting in uncertainty.

- *Implementation:* The controller implemented might not be the same as the one obtained by the design procedure. For example, often the design is performed in continuous time and implementation digital. The involved delays and approximations lead to uncertainty.

Classes of Uncertainty

- *Parametric uncertainty:* The model order and structure is assumed to be known. However, specific parameters that are real are uncertain in the model. Parametric uncertainty is quantified by assuming that the parameter lies in a certain region $[\alpha_{\min}, \alpha_{\max}]$.

- *Unmodeled or undermodeled dynamics uncertainty:* Here the model order and the structure is not certain. Such types of uncertainty results from either delibrate undermodeling or from a lack of physical understanding and unknown dynamics at higher frequecies.

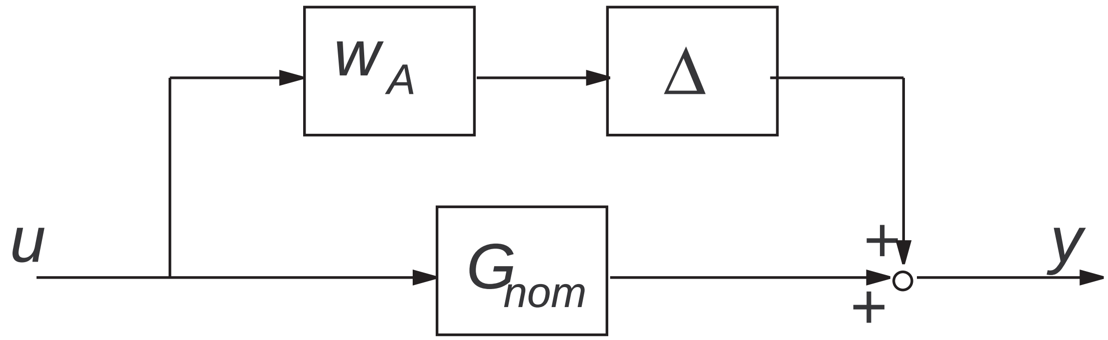
- *Lumped uncertainty:* This class of uncertainty can accommodate the above two types of uncertainty by lumping them into a single description.

Notation

- $\Pi_{LTI} :=$ { linear time invariant plants}.
- $\Pi_{LTV} :=$ { linear time varying plants}.
- $\Pi_{NL} :=$ { nolinear plants}.
- G_{nom} : The nominal plant assumed to be LTI.

We will identify Π to be Π_{LTI} unless otherwise stated. Also, we represent by G_p any element of Π; G_p is the *perturbed plant* in contrast to G_{nom} which is the nominal plant.

Typical Uncertainty Characterization



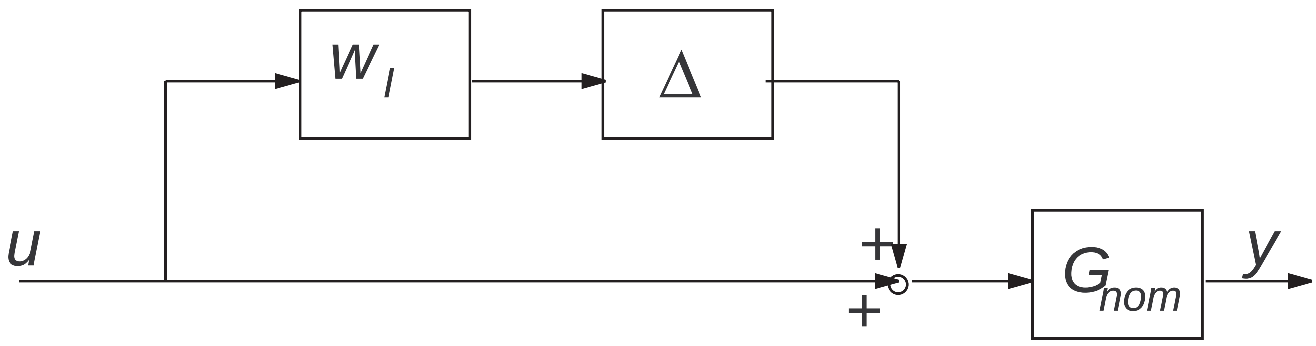
Additive uncertainty:

$$G_p = G_{nom} + w_A\Delta$$

- w_A is a weight usually chosen to be stable and minimum phase
- Δ is scaled to lie is a set. For example

$$\Delta \in \{\Delta \in \mathcal{H}_\infty | \|\Delta\|_{\mathcal{H}_\infty} \leq 1\}.$$

Multiplicative Uncertainty



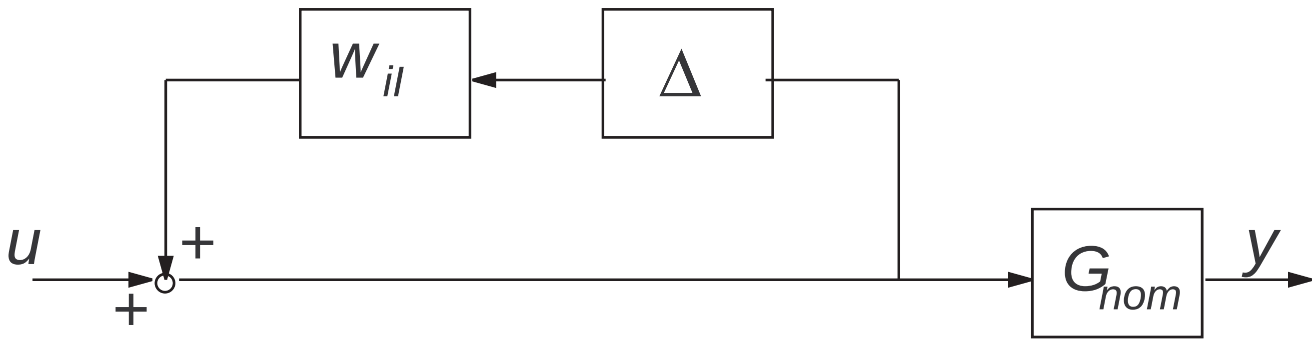
Multiplicative uncertainty:

$$G_p = G_{nom}(I + w_I \Delta)$$

- w_I is a weight usually chosen to be stable and minimum phase
- Δ is scaled to lie is a set. For example

$$\Delta \in \{\Delta \in \mathcal{H}_\infty | \|\Delta\|_{\mathcal{H}_\infty} \leq 1\}.$$

Inverse Multiplicative Uncertainty



Inverse Multiplicative uncertainty:

$$G_p = G_{nom}(I + w_{iI}\Delta)^{-1}$$

- w_{iI} is a weight usually chosen to be stable and minimum phase
- Δ is scaled to lie is a set. For example

$$\Delta \in \{\Delta \in \mathcal{H}_\infty | \|\Delta\|_{\mathcal{H}_\infty} \leq 1\}.$$

Parametric Uncertainty

The uncertain parameter is assumed to lie inside an interval $[\alpha_{\min}, \alpha_{\max}]$. Thus α can be represented by

$$\alpha = \overline{\alpha}(1 + r_\alpha \Delta)$$

where $\overline{\alpha} = \frac{\alpha_{\min}+\alpha_{\max}}{2}$, $r_\alpha = \frac{\alpha_{\min}-\alpha_{\max}}{\alpha_{\min}+\alpha_{\max}}$ and $\Delta \in [-1, 1]$.

Example 2. *(Gain uncertainty) Let*

$$\Pi = \{k_p G_0(s) | k_{\min} \leq k_p \leq k_{\max}\}$$

Define $\overline{k} = \frac{k_{\min}+k_{\max}}{2}$, $r_k = \frac{k_{\min}-k_{\max}}{k_{\min}+k_{\max}}$ *and* $\Delta \in [-1, 1]$. *Then*

$$G_p(s) = \underbrace{\overline{k} G_0(s)}_{G_{nom}(s)}(1 + r_k \Delta).$$

G_p *is in the multiplicative uncertainty form.*

Example 3. *(Time constant uncertainty) Let*

$$\Pi = \{\frac{1}{\tau_p s + 1} G_0(s) | \tau_{\min} \leq \tau \leq \tau_{\max}\}.$$

Define $\overline{\tau} = \frac{\tau_{\min}+\tau_{\max}}{2}$, $r_\tau = \frac{\tau_{\min}-\tau_{\max}}{\tau_{\min}+\tau_{\max}}$ *and* $\Delta \in [-1, 1]$. *Then*

$$G_p(s) = \frac{1}{\tau_p s + 1} G_0(s) = \frac{1}{\overline{\tau} s + \overline{\tau} r_\tau \Delta s + 1} G_0(s) = \underbrace{\frac{1}{\overline{\tau} s + 1} G_0(s)}_{G_{nom}(s)} (1 + w_{iI}(s)\Delta)^{-1};$$

where

$$w_{iI} = \frac{\overline{\tau} r_\tau s}{1 + \overline{\tau} s}.$$

G_p *is in the inverse multiplicative form.*

Example 4. *Consider a plant with an uncertain zero*

$$\Pi := \{\frac{s+a}{s^2+3s+1} | a_{\min} \leq a \leq a_{\max}\}.$$

- *Multiplicative uncertainty form*

  Define $\overline{a} = \frac{a_{\min}+a_{\max}}{2}$, $r_a = \frac{a_{\max}-a_{\min}}{2}$ *and* $\Delta \in [-1, 1]$. *This implies that*

  $$G_p(s) = \frac{s+a}{s^2+3s+1} = \frac{s+\overline{a}+r_a\Delta}{s^2+3s+1} = \underbrace{(\frac{s+\overline{a}}{s^2+3s+1})}_{G_{nom}(s)}(I + w_I\Delta)$$

  where $w_I = \frac{r_a}{s+\overline{a}}$.

- *Additive uncertainty form*

  Define $\overline{a} = \frac{a_{\min}+a_{\max}}{2}$, $r_a = \frac{a_{\max}-a_{\min}}{2}$ *and* $\Delta \in [-1, 1]$.

This implies that

$$G_p(s) = \frac{s+a}{s^2+3s+1} = \frac{s+\overline{a}+r_a\Delta}{s^2+3s+1} = \underbrace{\frac{s+\overline{a}}{s^2+3s+1}}_{G_{nom}(s)} + w_A\Delta$$

where $w_A = \frac{r_a}{s^2+3s+1}$.

Remarks

- Either multiplicative or additive uncertainty forms can be used to represent the uncertain set Π.
- Multiplicative uncertainty form represents *relative* error:

$$\frac{G_p - G_{nom}}{G_{nom}} = w_I \Delta.$$

- Additive uncertainty form represents the absolute model error:

$$G_p - G_{nom} = w_A \Delta.$$

Robust Control Oriented Modeling

The modeling suited for the robust control paradigm involves the following steps:

1. Obtain the model class Π. G_p is any particular element of Π.

2. Obtain the nominal model $G_{nom}(s)$.

3. Obtain the bound on the deviation of the actual behavior of the plant from the nominal behavior.

   - For the additive uncertainty characterization the deviation is given by the function
   $$\ell_A(\omega) = \max_{G_p \in \Pi} |G_p(j\omega) - G(j\omega)|.$$
   - For the multiplicative uncertainty characterization the deviation is given

by the function

$$\ell_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right|.$$

4. Obtain the weight that describes the deviation, that is, choose a rational weight ($w_A(s)$ and $w_I(s)$ for additive and multiplicative uncertain form respectively) that has low order stable and such that $|w_\alpha(j\omega)| \geq \ell_\alpha(j\omega)$ (where $\alpha = A$ or $\alpha = I$).

Obtaining a Model Class II

The following methods can be utilized

1. A model derived based on the understanding of the plant. The plant model for example could be derived based on physical principles and rough estimates on the parameters of the model can be derived.

   - Advantages: The resulting model is typically simple and captures the qualitative dynamics well.
   - Disadvantages: Not always possible or difficult to obtain. For example in the nanopositioning example the serpentine stage is quite intricate and obtaining a model of the system based on physical arguments is difficult.

2. Evaluate the frequency response of the system at various experimental conditions and obtain the frequency response repeated number of times. For example, if the frequency response is obtained about different bias voltages to the the piezo different plots are obtained for the nanopositioning

stage. Also, the gain at DC depends on the history of the applied voltage due to hysteresis. Also, the plant is slightly time varying due to creep and other effects.

- Advantages: Not hard to obtain as it does not involve much analysis.
- Disadvantages: The resulting model can be quite involved and might not capture the physics of the plant.

Obtaining a Nominal Model

Once the model class Π is obtained one has to choose a nominal model and the associated uncertainty has to be determined. The following approaches can be taken to identify the nominal model.

1. A simplified model obtained by ignoring delays and higher order dynamics. For example, if the model class was determined to be

$$G_p(s) = \frac{s+20}{(s+1)(0.1s+1)}e^{-\theta s}$$

with $\theta \in [\theta_{\min}, \theta_{\max}]$ then one can choose

$$G_{nom}(s) = \frac{20}{s+1}$$

The advantages are the simplicity of the nominal model that can lead to easier controller design. The disadvantage is the large uncertainty that might result.

2. If the model class is characterized by multiple parameters then choose the nominal model to the one with the parameters taken to be central values of the ranges involved. (see the examples derived earlier).

3. At every ω choose $G(j\omega)$ as the point on the Nyquist plot that leads to the smallest uncertainty. This leads to the smallest uncertainty however, it needs considerable effort, the resulting nominal plant can be of very high order and the nominal model might not capture the essential features of the system.

4. Typically a judicious combination of the above three methods provides the best alternative.

Determining the Uncertainty Bound

Once the nominal model is fixed, then the uncertainty bound $\ell(j\omega)$ has be determined. Note that for the additive uncertainty case the bound is defined as

$$\ell_A(j\omega) = \sup_{G_p \in \Pi} |G_p(j\omega) - G_{nom}(j\omega)| \forall \omega$$

whereas for multiplicative uncertainty form we have

$$\ell_I(j\omega) = \sup_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G_{nom}(j\omega)}{G_{nom}(j\omega)} \right| \forall \omega$$

It is evident that the above formulae cannot always be utilized to generate the bound mainly because the $\sup$ is over an infinite number of plants and it has to be evaluated over all $\omega \in R$. Different techniques are employed depending upon the data available.

Example 5. *(Model is known with parameters uncertain) In this case, one possible method of evaluating the bound is to first grid the parameter region and obtain the frequency plot for each parameter vector on the grid. Let* G_k *denote the* k^{th} *model. A grid is obtained on the frequency region* ω. *Let the corresponding frequency vector be* $\Omega = \{\omega_1, \ldots, \omega_n\}$. *For additive uncertainty*

$$\ell_A(j\omega_i) = \max_k |G_k(j\omega_i) - G_{nom}(j\omega_i)| \forall \omega_i \in \Omega$$

and in the case of multiplicative uncertainty we have

$$\ell_I(j\omega_i) = \max_k \left| \frac{G_p(j\omega_i) - G_{nom}(j\omega_i)}{G_{nom}(j\omega_i)} \right| \forall \omega_i \in \Omega.$$

We will obtain the multiplicative uncertainty description of the following class:

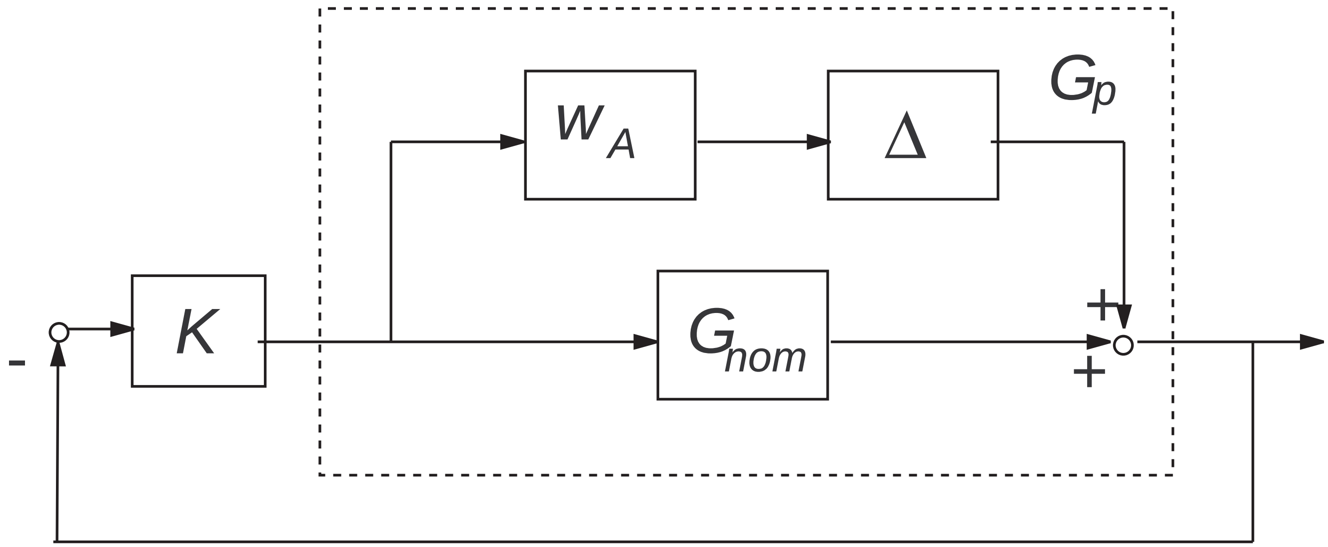
$$\Pi = \{\frac{k}{\tau s + 1} e^{-\theta s},\ 2 \le k, \theta, \tau \le 3\}.$$

The nominal model is chosen as

$$G_{nom}(s) = \frac{2.5}{2.5s + 1}.$$

The attached Matlab code does the appropriate gridding of the parameter space and the frequencies.

Robust Stability Condition for Additive Uncertainty



We will consider the following model class

$$\Pi := \{G(s) + w_A(s)\Delta(s) \;\; | \|\Delta\|_{\mathcal{H}_\infty} \leq 1\}.$$

where $w_A(s)$ is assumed to be a stable, proper rational transfer function. We will denote by $L = KG$ and by $L_p = KG_p$ where $G_p \in \Pi$.

Assumption 1. *We will assume that the nominal model $G(s)$ is such that the unity feedback configuration shown in the figure above (with $\Delta = 0$) is stable.*

Theorem 13. *The closed loop system shown in Figure is robustly stable (that is the for all $G_p \in \Pi$) if and only if*

$$\|w_A K S\|_{\mathcal{H}_\infty} < 1$$

where $S := (I + L)^{-1}$ is the sensitivity transfer function corresponding to the nominal plant.

Proof: By assumption we have that the with $\Delta = 0$ the closed loop system is stable. Let the number of encirclements of -1 by the Nyquist plot of L be N.

Note that as Δ and w_A are assumed to be stable, it follows that the number of poles in the right half plane of any $G_p K = GK + w_A K \Delta$ in Π is not greater than the number of rhp poles of $L = GK$.

($\Leftarrow$) We are given that the number of encirclements by the Nyquist plot of L is N. Suppose $\|w_A KS\|_{\mathcal{H}_\infty} < 1$. Thus

$$\begin{array}{rlcl} & \|w_A KS\|_{\mathcal{H}_\infty} & < & 1 \\ \Rightarrow & \|w_A KS\Delta\|_{\mathcal{H}_\infty} \leq \|w_A KS\|_{\mathcal{H}_\infty}\|\Delta\|_{\mathcal{H}_\infty} & < & 1 \text{ if } \|\Delta\|_{\mathcal{H}_\infty} \leq 1 \\ \Rightarrow & |(w_A K\Delta)(j\omega)| & < & |1+L(j\omega)| \text{ for all } \omega, \|\Delta\|_{\mathcal{H}_\infty} \leq 1 \\ \Rightarrow & |(L_p - L)(j\omega)| & < & |1+L(j\omega)| \text{ for all } \omega \end{array}$$

- Thus the distance of the Nyquist plot of L from -1 is greater than the distance of the perturbed open loop gain L_p from L. As L encircles -1 point N times L_p also encircles -1, N times.

- Note that if the number of rhp poles of L is P then as the nominal system is stable the number of rhp zeros Z of $1+L$ is zero we have $N = P$ (thus $Z - P = -N$ i.e N counterclockwise encirclements).

- Let Z_p and P_p denote the number of rhp poles zeros and poles of $1 + L_p$. As the number of counterclockwise encirclements of -1 point of L_p is N we

have $Z_p - P_p = -N$. Thus $Z_p = P_p - N$. However, we have already seen that $P \geq P_p$ as the weight w_A and Δ are stable. Thus
$Z_p = P_p - N \leq P - N = 0$.

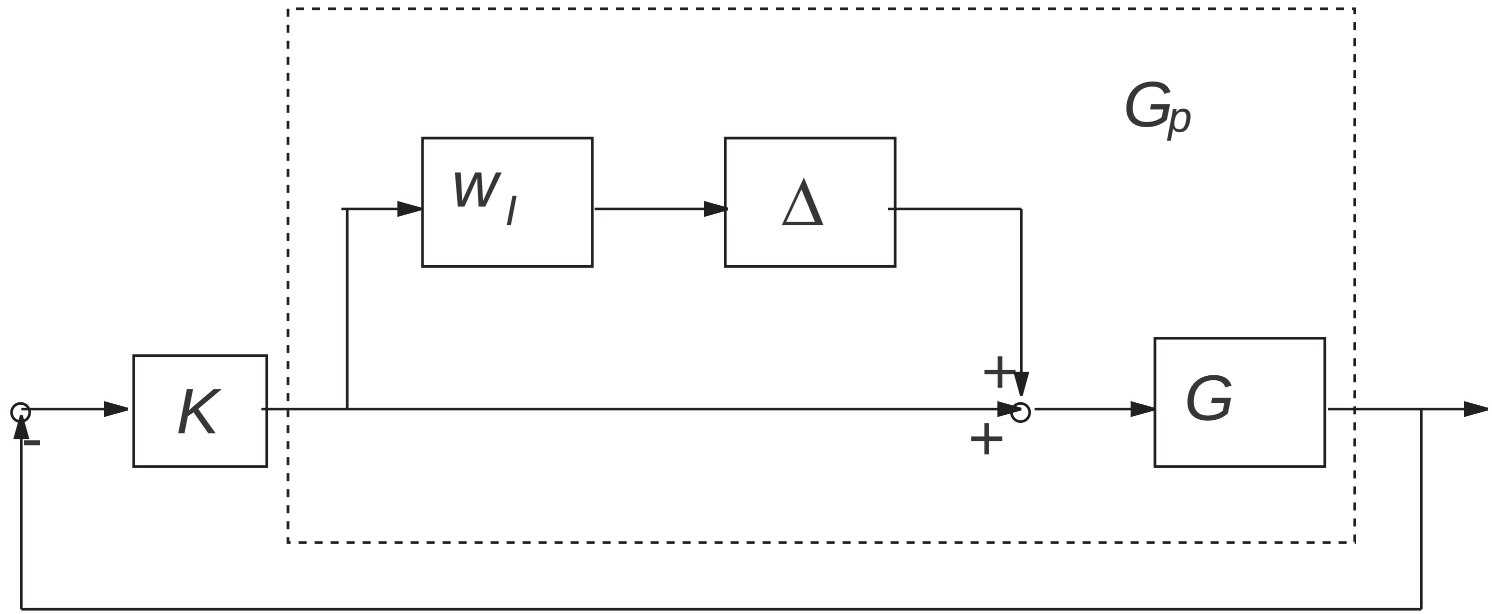
- This implies $Z_p = 0$. This in turn implies $P_p = P$ and that there are no unstable pole-zero cancellations in the product $G_p K$.

- Thus there are no unstable pole-zero cancellations in the product $G_p K$ and the number of counterclockwise encirclements of the -1 point on the complex plane is equal to the number of unstable poles of $G_p K$.

- From Theorem 5 the interconnection is stable.

We will not prove that $\|w_A K S\|_{\mathcal{H}_\infty} < 1$ is a necessary condition. ■

Robust Stability Condition for Multiplicative Uncertainty



We will consider the following model class

$$\Pi := \{G(s)(I + w_I(s)\Delta(s))| \ \ \|\Delta\|_{\mathcal{H}_\infty} \leq 1.\}$$

where $w_I(s)$ is assumed to be a stable, proper rational transfer function. We will denote by $L = KG$ and by $L_p = KG_p$ where $G_p \in \Pi$.

Assumption 2. *We will assume that the nominal model $G(s)$ is such that the unity feedback configuration shown in the figure above (with $\Delta = 0$) is stable.*

Theorem 14. *The closed loop system shown in Figure is robustly stable (that is the for all $G_p \in \Pi$) if and only if*

$$\|w_I T\|_{\mathcal{H}_\infty} < 1$$

where $T := L(I + L)^{-1}$ is the complimentary sensitivity transfer function corresponding to the nominal plant.

Proof: By assumption we have that the with $\Delta = 0$ the closed loop system is stable. Let the number of encirclements of -1 by the Nyquist plot of L be N.

Note that as Δ and w_I are assumed to be stable, it follows that the number of poles in the right half plane of any $L_p = G_p K = GK(1 + w_I \Delta)$ in Π is not greater than the number of rhp poles of $L = GK$.

($\Leftarrow$) We are given that the number of encirclements by the Nyquist plot of L is N. Suppose $\|w_I T\|_{\mathcal{H}_\infty} < 1$. Thus

$$\begin{array}{lcl} \|w_I T\|_{\mathcal{H}_\infty} & < & 1 \\ \Rightarrow \|w_I T\Delta\|_{\mathcal{H}_\infty} \leq \|w_I T\|_{\mathcal{H}_\infty}\|\Delta\|_{\mathcal{H}_\infty} & < & 1 \text{ if } \|\Delta\|_{\mathcal{H}_\infty} \leq 1 \\ \Rightarrow |(w_I G K \Delta)(j\omega)| & < & |1 + L(j\omega)| \text{ for all } \omega \in R \text{ if } \|\Delta\|_{\mathcal{H}_\infty} \leq 1 \\ \Rightarrow |(L_p - L)(j\omega)| & < & |1 + L(j\omega)| \text{ for all } \omega \in R \end{array}$$

- Thus the distance of the Nyquist plot of L from -1 is greater than the distance of the perturbed open loop gain L_p from L. As L encircles -1 point N times L_p also encircles -1, N times. Note that if the number of rhp poles of L is P then as the nominal system is stable the number of rhp zeros Z of $1 + L$ is zero we have $N = P$ (thus $Z - P = -N$ i.e N counterclockwise encirclements).

- Let Z_p and P_p denote the number of rhp poles zeros and poles of $1 + L_p$. As the number of counterclockwise encirclements of -1 point of L_p is N we have $Z_p - P_p = -N$. Thus $Z_p = P_p - N$.

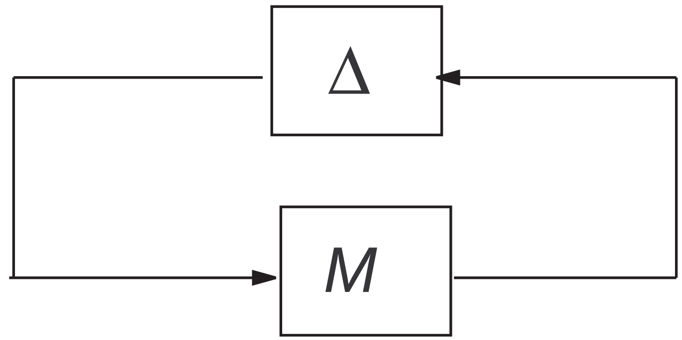
- However, we have already seen that $P \geq P_p$ as weight w_I and Δ are stable. Thus $Z_p = P_p - N \leq P - N = 0$. This implies $Z_p = 0$.

- Thus there are no unstable pole-zero cancellations in the product $G_p K$ and the number of counterclockwise encirclements of the -1 point on the complex plane is equal to the number of unstable poles of $G_p K$.

- From Theorem 5 the interconnection is stable.

We will not prove that $\|w_I T\|_{\mathcal{H}_\infty} < 1$ is a necessary condition. ■

The $M - \Delta$ Configuration



Theorem 15. *Consider the interconnection depicted in the Figure above where M and Δ are two LTI stable causal systems such that $\|\Delta\|_{\mathcal{H}_\infty} \leq 1.$ Then the interconnection is stable if and only if $\|M\|_{\mathcal{H}_\infty} < 1.$*

Proof: It follows from the Nyquist criterion that as M and Δ are stable (no rhp poles) the unity feedback interconnection of Figure is stable if and only if $|1 + M(j\omega)\Delta(j\omega)|$ does not encircle or touch the point $0.$

- Thus robust stability is achieved if and only if

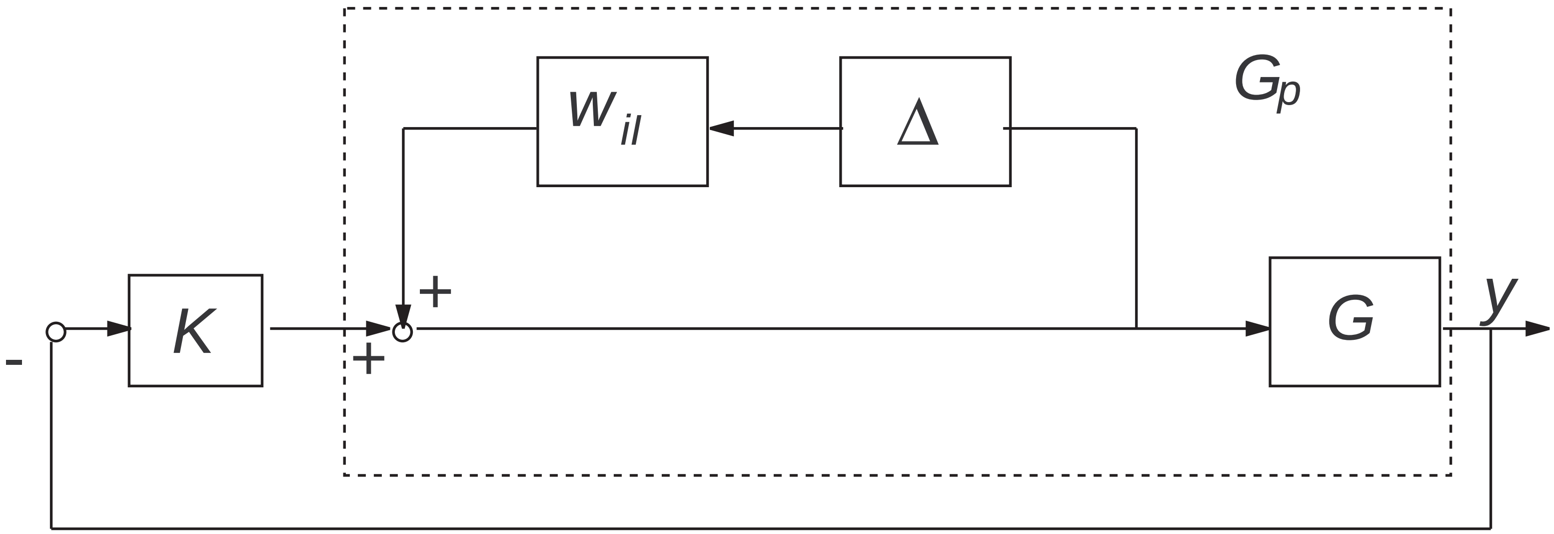
$$|1 + M(j\omega)\Delta(j\omega)| > 0,\ \forall\omega,\ \forall\Delta \text{ such that } \|\Delta\|_{\mathcal{H}_\infty} \leq 1. \qquad (2)$$

- If $\|M\|_{\mathcal{H}_\infty} < 1$ then $1 - |M(j\omega)|\ |\Delta(j\omega))| > 0$ for any $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$ (as $|M(j\omega)|\ |\Delta(j\omega))| < 1$). Thus $|1 + M(j\omega)\Delta(j\omega)| > 0$ for all ω. Thus robust stability is ensured if $\|M\|_{\mathcal{H}_\infty} < 1$.

- Suppose $\|M\|_{\mathcal{H}_\infty} \geq 1$ then we can construct a Δ such that Δ is a stable proper transfer function with $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$ and there exists an ω where $1 + M(j\omega)\Delta(j\omega) = 0$. This would violate the condition (2) and thus there is no robust stability.

$\blacksquare$

Robust Stability Condition for Inverse Multiplicative Uncertainty



We will consider the following model class

$$\Pi := \{G(s)(I + w_{iI}(s)\Delta(s))^{-1} \ | \ \|\Delta\|_{\mathcal{H}_\infty} \leq 1.\}$$

where $w_{iI}(s)$ is assumed to be a stable, proper rational transfer function. We will denote by $L = KG$ and by $L_p = KG_p$ where $G_p \in \Pi$.

Assumption 3. *We will assume that the nominal model $G(s)$ is such that the unity feedback configuration shown in the figure above (with $\Delta = 0$) is stable.*

Theorem 16. *The closed loop system shown in Figure is robustly stable (that is the for all $G_p \in \Pi$) if and only if*

$$\|w_{iI}S\|_{\mathcal{H}_\infty} < 1$$

where $S := (I + L)^{-1}$ is the sensitivity transfer function corresponding to the nominal plant.

Proof: We will apply Theorem 15 to obtain the result. Note that the equivalent M *seen* by Δ is

$$M = \frac{w_{iI}}{1 + GK} = w_{iI}S.$$

Thus robust stability holds if and only if

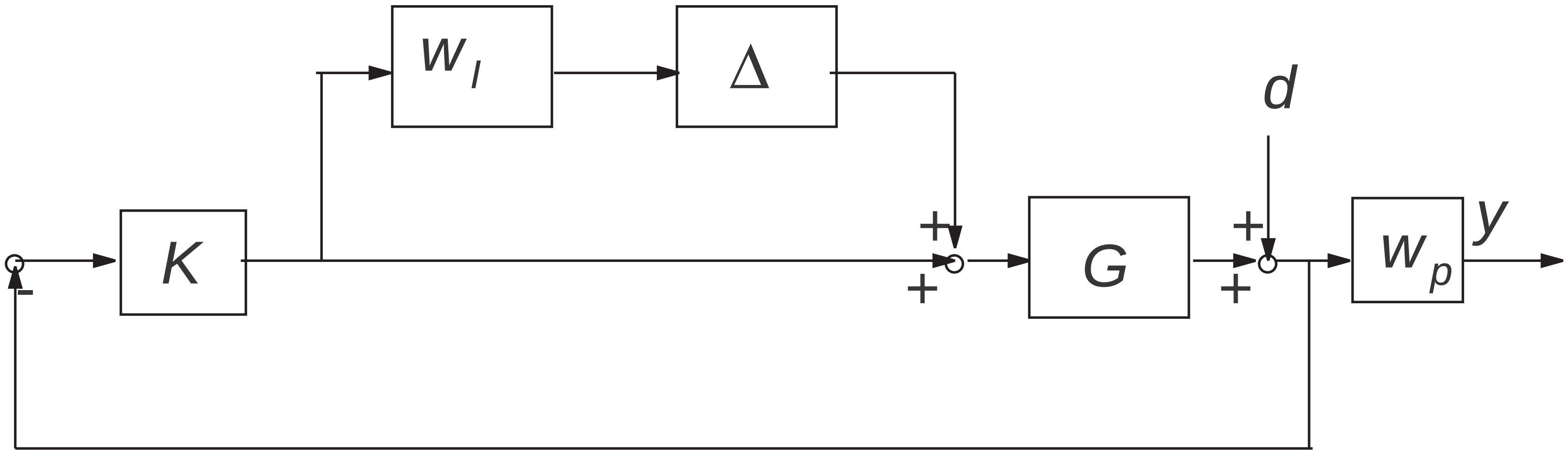
$$\|w_{iI}S\|_{\mathcal{H}_\infty} < 1.$$

■

Note that the other robust stability conditions could have been derived in this manner.

Robust Performance for SISO systems

Robust Performance



In the robust performance problem the following are the objectives

- Robust stability
- Performance for all the plants in the model class Π.

Consider the feed-back loop shown where the plant class is described by multiplicative uncertainty. The robust stability criteria was determined to be

$$RS \Leftrightarrow \|w_I T\|_{\mathcal{H}_\infty} < 1.$$

The performance desired in the above setup is that of tracking and/or disturbance rejection (which are the same if $G_d = I$. Thus the performance requirement is

$$\|w_p S_p\|_{\mathcal{H}_\infty} < 1$$

where

$$S_p = (I + L_p)^{-1}$$

with $L_p = G_p K$ where

$$G_p \in \Pi := \{G(1 + w_I \Delta)| \ \ \|\Delta\|_{\mathcal{H}_\infty} \leq 1\|.$$

The performance condition translates to the condition that for all $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$

$$\|w_p \frac{1}{1 + GK(1 + w_I \Delta)}\|_{\mathcal{H}_\infty} < 1 \Leftrightarrow \|\frac{w_p S}{1 + w_I T \Delta}\|_{\mathcal{H}_\infty} < 1 \qquad (3)$$

We summarize the above observations as a Lemma.

Lemma 2. *Necessary and sufficient conditions for robust performance are*

1. $\|w_I T\|_{\mathcal{H}_\infty} < 1.$

2. $\|\frac{w_p S}{1+w_I T\Delta}\|_{\mathcal{H}_\infty} < 1,\ \forall$ *stable* Δ *with* $\|\Delta\|_{\mathcal{H}_\infty} \leq 1.$

Theorem 17. *A necessary and sufficient condition for robust performance for the interconnection in the Figure is*

$$\||w_p S| + |w_i T|\|_{\mathcal{H}_\infty} < 1.$$

Proof: ($\Leftarrow$) Let

$$\||w_p S| + |w_i T|\|_{\mathcal{H}_\infty} < 1. \tag{4}$$

Then the robust stability requirement $\|w_I T\|_{\mathcal{H}_\infty} < 1$ is satisfied. Let Δ be fixed with $\|\Delta\|_{\mathcal{H}_\infty} \leq 1.$

From (4) it follows that

- for all ω, $|w_pS| + |w_IT| < 1$ which implies $|w_pS| < 1 - |w_IT| < 1 - |w_IT||\Delta| < |1 + w_IT\Delta|$
- thus
$$\frac{|w_pS|}{|1 + w_IT\Delta|} < 1 \; \forall \; \omega.$$
Thus all conditions of Lemma 2 are satisfied and robust performance follows.

($\Rightarrow$) Suppose there exists a ω_0 such that

$$|(w_pS)(j\omega_0)| + |(w_IT)(j\omega_0)| > 1.$$

- (Case 1) If $|(w_IT)(j\omega_0)| \geq 1$ then

$$\|w_IT\|_{\mathcal{H}_\infty} \geq |(w_IT)(j\omega_0)| > 1$$

and therefore there is no robust stability and therefore no robust performance. Thus the proof is complete.

- (Case 2) Suppose $|(w_I T)(j\omega_0)| < 1$.

  - ⋆ Note that as $|(w_p S)(j\omega_0)| + |(w_I T)(j\omega_0)| > 1$, we have

$$\begin{array}{rcl} |(w_p S)(j\omega_0)| & > & 1 - |(w_I T)(j\omega_0)| \\ \Rightarrow \quad \frac{|(w_p S)(j\omega_0)|}{1-|(w_I T)(j\omega_0)|} & \geq & 1 \end{array}$$

  - ⋆ Construct a transfer function Δ that is stable with $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$ such that

$$|1 - |(w_I T)(j\omega_0)|| = |(1 + w_I T\Delta)(j\omega_0)|.$$

    This is indeed possible and is left as an exercise

⋆ It follows that

$$\begin{aligned}\|\tfrac{w_pS}{1+w_IT\Delta}\|_{\mathcal{H}_\infty} &\geq \frac{|(w_pS)(j\omega_0)|}{|(1+w_IT\Delta)(j\omega_0)|}\\ &= \frac{|(w_pS)(j\omega_0)|}{|(1-(|w_IT\Delta)(j\omega_0)|)|}\\ &\geq 1\end{aligned}\;.$$

From Lemma 2 we have that there is no robust performance.

This completes the proof.

■

Summary

- Nominal Performance $\Leftrightarrow \|w_p S\|_{\mathcal{H}_\infty} < 1.$
- Robust Stability $\Leftrightarrow \|w_I T\|_{\mathcal{H}_\infty} < 1.$
- Nominal performance and robust stability $\Leftrightarrow \| \max(|w_p S|, |w_I T|)\|_{\mathcal{H}_\infty} < 1$ (follows from the above two conditions).
- Robust Performance $\Leftrightarrow \| |w_p S| + |w_I T| \|_{\mathcal{H}_\infty} < 1.$

It can be shown that

$$\frac{1}{\sqrt{2}}(|w_p S| + |w_I T|) \leq (|w_p S|^2 + |w_I T|^2)^{\frac{1}{2}} \leq \sqrt{2}(|w_p S| + |w_I T|)$$

and

$$\max(|w_pS|, |w_IT|) \leq (|w_pS|^2 + |w_IT|^2)^{\frac{1}{2}} \leq 2\max(|w_pS|, |w_IT|).$$

Thus the following lemma holds:

Lemma 3. *The following hold:*

1. $\| \max(|w_pS|, |w_IT|)\|_{\mathcal{H}_\infty} \leq \left\| \begin{array}{c} w_pS \\ w_IT \end{array} \right\|_{\mathcal{H}_\infty} \leq 2\| \max(|w_pS|, |w_IT|)\|_{\mathcal{H}_\infty}$

2. $\frac{1}{\sqrt{2}}\||(w_pS| + |w_IT)|\|_{\mathcal{H}_\infty} \leq \left\| \begin{array}{c} w_pS \\ w_IT \end{array} \right\|_{\mathcal{H}_\infty} \leq \sqrt{2}\ \|(|w_pS| + |w_IT|)\|_{\mathcal{H}_\infty}.$

Proof: Follows from the fact that

$$\left\| \begin{array}{c} w_pS \\ w_IT \end{array} \right\|_{\mathcal{H}_\infty} = \sup_\omega(|w_pS|^2 + |w_IT|^2)^{\frac{1}{2}}.$$

■

Thus the conclusion is that be solving an appropriately scaled *stacked* $\mathcal{H}_\infty$ problem one can achieve the objectives of robust performance. Note that we have employed the stacked framework to obtain robust stability and *nominal* performance for the nanopositioning example.

Optimal Controller Synthesis For SISO Systems

Coprime Factors: Single Input Single Output Case

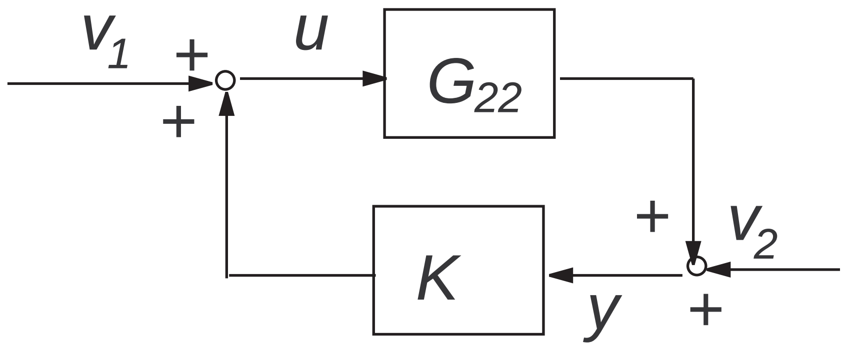


Figure 1:

- Let the plant G_{22} be given by $G_{22} = \frac{N(s)}{M(s)}$ and $K = \frac{Y(s)}{X(s)}$ where $N(s),\ M(s),\ X(s),$ and $Y(s)$ are polynomials in s. We assume there are no common factors in the ratios being formed.

- We assume that G_{22} and K are rational functions of s.

Notice that for the SISO positive feedback configuration shown in the figure we have

$$1 - G_{22}K = 1 - \frac{NY}{MX} = \frac{MX - NY}{MX}.$$

By the Nyquist stability criterion the feedback interconnection is stable if and only if

$$MX - NY$$

has no zeros in the right half plane.

In other words we need

$$R := (MX - NY)^{-1}$$

to be stable transfer function for the interconnection to be stable.

Note that K can be written as $K = \frac{Y_1}{X_1}$ where

$$Y_1 := YR \text{ and } X_1 = XR$$

and

$$MX_1 - NY_1 = 1.$$

Definition 13. *We say two rational stable functions X and Y are coprime over stable systems if they do not have common unstable factors.*

We will use the term coprime to mean coprime over stable systems.

We make the above arguments precise with the the following lemma.

Lemma 4. *Let $G_{22} = \frac{N}{M}$ be the plant with N and M being coprime. K is a stabilizing controller for the feedback interconnection shown in Figure 1 if and only if there exist stable coprime factors Y and X such that $K = \frac{Y}{X}$ satisfying*

$$MX - NY = 1.$$

Proof: From the Nyquist stability criterion (Theorem 5) the closed-loop system is stable if and only if

1. There is no rhp pole zero cancellation while forming the product $GK = \frac{NY}{MX}$

2. $S = (I - GK)^{-1} = \frac{1}{1-\frac{NY}{MX}} = \frac{MX}{MX-NY}$ is stable.

We will first show that the above two conditions are equivalent to the condition that $MX - NY$ have no rhp zeros.

Indeed let $MX - NY$ have no rhp zeros. Clearly then MX and NY have no common rhp zeros. Thus there can be no rhp pole zero cancellation while forming the product GK. Also, as $MX - NY$ has no rhp zeros $(MX - NY)^{-1}$ is stable. This implies $S = \frac{MX}{MX-NY}$ is stable. Thus the two conditions for stability of closed loop map are satisfied.

Now, suppose (1) and (2) are met. Then MX and NY have no common rhp zeros and $\frac{MX}{MX-NY}$ is stable. Suppose that $MX - NY$ has a rhp zero at z. Then $M(z)X(z) = N(z)Y(z)$. Also as $\frac{MX}{MX-NY}$ is stable it has to be true that the rhp zero z of the denominator $MX - NY$ be cancelled by the rhp zero of the numerator MX. Thus $M(z)X(z) = 0$. This would imply $M(z)X(z) - N(z)Y(z) = 0$ and thus $MX - NY$ has a rhp zero at z which implies that MX and NY have a common rhp zero. This would contradict (1). Thus $MX - NY$ has no rhp zeros.

Thus, it follows that stability of the closed loop system is equivalent to $MX - NY$ having no zeros in the rhp. Or equivalently closed loop system is stable if and only if $R = (MX - NY)^{-1}$ is stable. Clearly $K = Y/X = \frac{YR}{XR} = Y_1/X_1$ where $Y_1 = YR$ and $X_1 = XR$ are coprime. Furthermore

$$MX_1 - NY_1 = (MX - NY)R = 1.$$

This proves the theorem. $\blacksquare$.

Parametrization of Stabilizing Controllers for SISO Systems

In the SISO case the derivations of the class of stabilizing controllers is straightforward.

Theorem 18. *Let* $P = \frac{N}{M}$ *and let* $K = \frac{Y_1}{X_1}$ *be a stabilizing controller with* N *and* M *coprime and* Y_1 *and* X_1 *coprime with*

$$MX_1 - NY_1 = 1.$$

Then all stabilizing controllers are given by

$$K = \frac{Y_1 - MQ}{X_1 - NQ}$$

where Q *is any stable rational function.*

Proof:: $(\Leftarrow)$ Let

$$K = \frac{Y_1 - MQ}{X_1 - NQ}.$$

It follows that

$$M(X_1 - NQ) - N(Y_1 - MQ) = MX_1 - NY_1 = 1.$$

From Lemma 4 it follows that the closed-loop is stable.

$(\Rightarrow)$ Suppose K is a stabilizing controller. We infer from Lemma 4 that $K = \frac{Y}{X}$ with Y and X such that

$$MX - NY$$

is stable Define Q to satisfy the relation

$$\frac{Y}{X} = \frac{Y_1 - MQ}{X_1 - NQ}$$

which holds if

$$Q = -\frac{X_1 Y - Y_1 X}{MX - NY}$$

which is stable.

This proves the theorem.

■

Let K be a stabilizing controller. Then it follows that there exists a stable Q such that $K = \frac{Y_1 - MQ}{X_1 - NQ}$.

$$\begin{aligned}
S &= \frac{1}{1-GK} \\
&= \frac{1}{1-\frac{N}{M}\frac{Y_1-MQ}{X_1-NQ}} \\
&= \frac{M(X_1-NQ)}{M(X_1-NQ)-N(Y_1-MQ)} \\
&= \frac{MX_1-MNQ}{MX_1-MNQ-NY_1+NMQ} \\
&= MX_1 - MNQ
\end{aligned}$$

It follows that

$$T = 1 - S = 1 - MX_1 + MNQ = NY_1 + MNQ.$$

and

$$KS = \frac{Y_1 - MQ}{X_1 - NQ} M(X_1 - NQ) = M(Y_1 - MQ).$$

Note that the stacked $\mathcal{H}_\infty$ problem is given by

$$\begin{aligned} \mu &= \inf_{K \text{ stabilizing}} \left\| \begin{array}{l} w_P S(K) \\ w_T T(K) \\ w_u KS \end{array} \right\|_{\mathcal{H}_\infty} \\ &= \inf_{Q \text{ stable}} \left\| \begin{array}{l} w_P M(X_1 - NQ) \\ w_T N(Y_1 + MQ) \\ w_u M(Y_1 - MQ) \end{array} \right\|_{\mathcal{H}_\infty} \end{aligned}$$

Generalized Plant

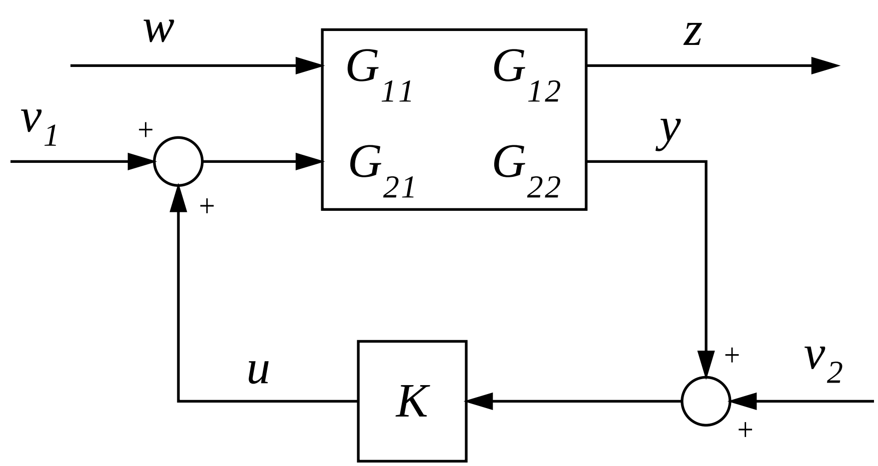


Figure 2:

Many control design issues can be cast into the framework shown in Figure 2.

Lets assume that $v_1 = v_2 = 0$. Then

$$\begin{aligned} z &= G_{11}w + G_{12}u \\ y &= G_{21}w + G_{22}u \end{aligned}$$

When we substitute $u = Ky$ we have
$y = G_{21}w + G_{22}Ky \Rightarrow\; y = (I - G_{22}K)^{-1}G_{21}w$. Thus

$$\begin{aligned} z &= [G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}]w \\ &= [G_{11} + G_{12}M(Y_1 - MQ)G_{21}]w \end{aligned} \quad .$$

The map bewteen z and w is *affine linear* in the parameter Q.

Multiple Input Multiple Output Systems

Linear Systems

We present notions of stability, causality and well-posedness of interconnections of systems.

Notation

- Signals

$$L^n = \{x : x = (x_1, x_2, \ldots, x_n) \text{ with } x_i \in L\}.$$

- Signal-norms

  For any x in L^n let

$$\begin{aligned}
\|x\|_p &= \left(\int_{-\infty}^{\infty} \sum_{i=1}^{n} |x_i(t)|^p \right)^{\frac{1}{p}} \quad 1 \leq p < \infty \text{ and} \\
\|x\|_\infty &= \sup_t \max_i |x_i(t)|
\end{aligned}$$

- Signal Space Let

$$L_p^n = \{x | x \in L^n, \|x\|_p < \infty\}$$

- Let $L_p^{m\times n}$ denote the spaces of $m \times n$ matrices with each element of the matrix in L_p.

Truncation Operator

- Let P_τ denote the truncation operator on $L^{m\times n}$ which is defined by

$$\begin{array}{rcll} P_\tau(x(t)) & = & x(t) & \text{if } t \le \tau \\ & = & 0 & \text{if } t > \tau \end{array}.$$

Shift Operator

- Let S denote the shift map from L^n to L^n defined by

$$S_\tau(x(t)) = x(t-\tau).$$

Causal Systems

Definition 14. [Causality] • *A linear map $\mathcal{T} : L^n \to L^m$ is said to be causal if*

$$P_t\mathcal{T} = P_t\mathcal{T}P_t \text{ for all } t.$$

Time Invariant Systems

Definition 15. [Time invariance]

- *A map $\mathcal{T} : L^n \rightarrow L^m$ is time invariant if $S_\tau \mathcal{T} = \mathcal{T} S_\tau$ where S_τ is the shift operator.*

Induced Norms

Let $\mathcal{T}$ be a linear map from $(\mathcal{S}_p^n, \|.\|_p)$ to $(\mathcal{S}_p^m, \|.\|_p)$. The p-induced norm of $\mathcal{T}$ is defined as

$$\|\mathcal{T}\|_{p-ind} := \sup_{\|x\|_p \neq 0} \frac{\|\mathcal{T}x\|_p}{\|x\|_p}.$$

Input-output Stability

Definition 16. [Stability] *A linear map $\mathcal{T} : (X, \|.\|_X) \to (Y, \|.\|_Y)$ is said to be stable if it is bounded. That is $\mathcal{T}$ is stable if*

$$\|\mathcal{T}\| = \sup_{x \neq 0} \frac{\|\mathcal{T}x\|_Y}{\|x\|_X} = M < \infty.$$

Note that

- $\mathcal{T} : (L_p^n, \|.\|_p) \to (L_p^m \|.\|_p)$ is said to be L_p stable if it is a bounded operator.

Convolution Maps

Definition 17. [Convolution maps] *$\mathcal{T} : L^n \rightarrow L^m$ is linear, time invariant, convolution map if and only if $y = \mathcal{T} u$ is given by*

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} & \ldots & \mathcal{T}_{1n} \\ \mathcal{T}_{21} & \mathcal{T}_{22} & \ldots & \mathcal{T}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{T}_{m1} & \mathcal{T}_{m2} & \ldots & \mathcal{T}_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

where $y = (y_1\ y_2,\ \ldots,\ y_m) \in L^m$ and $u = (u_1,\ u_2,\ \ldots,\ u_n) \in L^n$, $\mathcal{T}_{ij} : L \rightarrow L$ is described by

$$(\mathcal{T}_{ij} x)(t) = \int_{-\infty}^{\infty} x(\tau) T_{ij}(t-\tau) d\tau$$

where $T_{ij}(t)$ termed the impulse response of the system $\mathcal{T}$.

Further $\mathcal{T}$ is also causal if $T_{ij}(t) = 0$ for all $t < 0$ and thus for any $x \in L$ we *have*

$$(\mathcal{T}_{ij}x)(t) = \int_0^\infty x(\tau)T_{ij}(t-\tau)d\tau$$

The operation given by Equation is often written as $y_i = T_{ij} * u_j$.

Note that *any* linear time invariant system can be described as a convolution map via its impulse response.

Transforms

Definition 18. • *For a linear, time invariant, causal, convolution map $T : L^n \rightarrow L^m$ the s-transform of T is defined as*

$$\hat{T}(s) := \int_0^\infty T(t) e^{-st} dt,$$

where

$$T(t) := \begin{pmatrix} T_{11}(t) & T_{12}(t) & \dots & T_{1n}(t) \\ T_{21}(t) & T_{22}(t) & \dots & T_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ T_{m1}(t) & T_{m2}(t) & \dots & T_{mn}(t) \end{pmatrix}.$$

It can be shown that $\hat{T}$ is analytic inside the open right half plane (open unit disc) and continuous on the boundary if the matrix $\{T(t)\} \in L_1^{m\times n}(\ell_1^{m\times n})$.

Transfer Function

Definition 19. [Rational, proper transfer functions] *Let $g(s) = \frac{n(s)}{d(s)}$ where n and d are polynomials in s. Then $g(s)$ is said to be a rational transfer function. Furthermore if $deg(d(s)) \geq deg(n(s))$ then $g(s)$ is a proper transfer function. If $deg(d(s)) = deg(n(s))$ then $g(s)$ is a bi-proper transfer function. If $deg(d(s)) > deg(n(s))$ then $g(s)$ is a strictly proper transfer function.*

Example 6. *e^{-s} is not a rational transfer function. $\frac{s}{s+1}$ is a strictly proper rational function. $\frac{s+2}{s+1}$ is a bi-proper transfer function.*

Note that a rational function $g(s)$

- is proper if and only if $g(\infty) = d$ a constant.
- is strictly proper if and only if $g(\infty) = 0$ a constant.
- is bi-proper if and only if $g(\infty) = d \neq 0$ where d is a constant.

Also note that every proper rational function can be written as

$$g(s) = g(\infty) + g_{sp}(s)$$

where $g_{sp}(s)$ is strictly proper (example: $\frac{s+2}{s+1} = 1 + \frac{1}{s+1}$).

Finite Dimensional Systems: Proper Transfer Matrices

Definition 20. [Finite dimensional system]

- *If the s-transform of any linear, time invariant, causal, map $T : L^n \to L^m$ is such that $\hat{T}_{ij}(s)$ is a proper transfer function then T represents a finite dimensional system. $\hat{T}(s)$ is said to be a proper transfer matrix. If every entry $\hat{T}_{ij}(s)$ is a strictly proper transfer function then $\hat{T}(s)$ is a strictly proper transfer matrix. If $\hat{T}$ is square with both $\hat{T}(s)$ and $\hat{T}^{-1}(s)$ being proper then $\hat{T}(s)$ is a bi-proper transfer matrix.*

We use the term FDLTIC as an abbreviation for finite-dimensional, linear, time invariant, causal. Note that FDLTIC maps are characterized by proper transfer matrices.

Analogous to the scalar case, any proper transfer matrix $G(s)$ can be written as

$$G(s) = G(\infty) + G_{sp}(s)$$

where G_{sp} is strictly proper transfer matrix.

Coprime Factors of Transfer Functions

Definition 21. [Coprime Factors] *Consider a transfer function $g(s) = n(s)/d(s)$ where $n(s)$ and $d(s)$ are polynomials in s. If $n(s)$ and $d(s)$ have no common factors then we say that $n(s)$ and $d(s)$ are coprime and $g = nd^{-1}$ is a coprime factorization of g.*

Example 7. *Let $g(s) = \frac{1}{s-1}$. Then $n = 1$ and $d = (s-1)$ gives a coprime factorization of g. Note that $g(s) = \frac{(s-2)}{s^2-3s+2}$. However, $n = (s-2)$ and $d = s^2 - 3s + 2$ is not a coprime factorization because of the common factor $(s-2)$.*

Degree of Transfer Functions

Definition 22. [Degree of a transfer function] *Given any proper transfer function* $g(s) = \frac{n(s)}{d(s}$ *let* $r(s)$ *be the greatest common divisor of the polynomials* $n(s)$ *and* $d(s)$. *Then* $n(s) = n_m(s)r(s)$ *and* $d(s) = d_m(s)r(s)$. *Clearly* $n_m(s)$ *and* $d_m(s)$ *will have no common divisor and therefore are coprime and* $g(s) = \frac{n_m(s)r(s)}{d_m(s)r(s)}$.

- $d_m(s)$ *is called the characteristic polynomial of* $g(s)$.
- *The degree of* $g(s)$ *is the degree of* $d_m(s)$.

Example 8. *Let* $g(s) = \frac{s^2-1}{4(s^3-1)} := \frac{n}{d}$. *The gcd of* $n(s)$ *and* $d(s)$ *is* $s - 1 =: r(s)$. *Thus* $n_m = s + 1$ *and* $d_m = 4s^2 + 4s + 1$ *with* d_m *being the characteristic polynomial. The degree of* $g(s)$ *is thus two (not three).*

Characterization of Coprimeness of Polynomials

Theorem 19. [Aryabhatta, Bezout, Diophantine] *Polynomials $n(s)$ and $d(s)$ are coprime if and only if there exist polynomials $n_c(s)$ and $d_c(s)$ such that*

$$n(s)n_c(s) + d(s)d_c(s) = 1.$$

Proof:($\Leftarrow$) Suppose there exist polynomials $n_c(s)$ and $d_c(s)$ such that

$$n(s)n_c(s) + d_m(s)d_c(s) = 1. \tag{5}$$

Also assume that $n(s) = n_m(s)r(s)$ and $d(s) = d_m(s)r(s)$ where $r(s)$ is a nonconstant polynomial, $d_m(s),\ n_m(s)$ are polynomials. Then we have from (5) that

$$r(s)[n_m(s)n_c(s) + d_m(s)d_c(s)] = 1.$$

This would imply that product of two non-trivial (i.e.nonconstant) polynomials is 1 which is not possible. This proves that $n(s)$ and $d(s)$ are coprime.

$(\Rightarrow)$ Assume that $n(s)$ and $d(s)$ are coprime. Then using the Euclidean Algorithm one can construct polynomials $n_c(s)$ and $d_c(s)$ such that

$$n(s)n_c(s) + d(s)d_c(s) = 1.$$

$\blacksquare$

Poles of a Transfer Matrix

Definition 23. [Characteristic polynomial, poles] *The characteristic polynomial of a transfer matrix $G(s)$ is the least common denominator of all minors of $G(s)$ where the minors are reduced to be coprime transfer functions. The degree of $G(s)$ is the degree of the characteristic polynomial. The poles are the roots of the characteristic polynomial.*

Example 9. *Consider the transfer matrix*

$$G(s) = \frac{1}{1.25(s+1)(s+2)} \begin{pmatrix} s-1 & s \\ -6 & s-2 \end{pmatrix}.$$

The minors of order 1 are the four elements of the transfer matrix $G(s)$ all of which have the same denominator $(s+1)(s+2)$. The minor of order two is the determinant of the matrix:

$$det(G(s)) = \frac{(s-1)(s-2)+6s}{1.25^2(s+1)^2(s+2)^2} = \frac{1}{1.25^2(s+1)(s+2)}.$$

Note that we have reduced the second order minor to be coprime. The lcd of all the minors is $(s+1)(s+2)$ *and thus the characteristic polynomial is given by*

$$\phi(s) = (s+1)(s+2).$$

The poles are given by $s = -1$ *and* $s = -2$.

Normal Rank of a Transfer Matrix

Definition 24. [Normal rank] *The normal rank of a transfer matrix $G(s)$ is the maximum rank of the transfer matrix over the variable s.*

Example 10. *Consider*

$$G(s) = \frac{1}{s+2}\begin{pmatrix} s-1 & 0 \\ 0 & 2(s-2) \end{pmatrix}.$$

The maximum rank of this matrix is two even though at $s = 1$ and $s = 2$ the rank drops to one. Thus the normal rank of the transfer matrix is two.

Zeros of a Transfer Matrix

Definition 25. [Zeros] *z_i is a zero of a transfer matrix $G(s)$ if rank of $G(z_i)$ is less than the normal rank of $G(s)$.*

Example 11. *Consider*

$$G(s) = \frac{1}{s+2}\begin{pmatrix} s-1 & 0 \\ 0 & 2(s-2) \end{pmatrix}.$$

Thus the normal rank of the transfer matrix is two. The rank of this matrix at $s=1$ and $s=2$ drops to one. Thus these are the zeros of the transfer matrix.

Unimodular Matrices

Definition 26. [Unimodular matrices] *A square polynomial matrix function $\hat{P}(\lambda) = P(0) + P(1)\lambda + \ldots + P(k)\lambda^k$, is said to be unimodular if the determinant of $\hat{P}(\lambda)$ is a non-zero constant independent of λ.*

Theorem 20. *Let $\hat{T}(\lambda)$ be a $m \times n$ matrix of rational functions of λ (a function is a rational function of λ if it can be written as a ratio of two polynomials of λ). Then there exist $\hat{L}$, $\hat{U}$ and $\hat{M}$ such that $\hat{T} = \hat{L}\hat{M}\hat{U}$ where $\hat{L}$ and $\hat{U}$ are unimodular with appropriate dimensions and $\hat{M}$ has the structure*

$$\hat{M} = \begin{pmatrix} \frac{\hat{\epsilon}_1}{\hat{\psi}_1} & & & 0 & \ldots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \frac{\hat{\epsilon}_r}{\hat{\psi}_r} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

$\{\hat{\epsilon}_i, \hat{\psi}_i\}$ *are coprime (that is they do not have any common factors) monic (leading coefficient is one) polynomials, which are not identically zero for all* $i = 1, \ldots, r$ *with the following divisibility property:* $\hat{\epsilon}_i(\lambda)$ *divides* $\hat{\epsilon}_{i+1}(\lambda)$ *without remainder and* $\hat{\psi}_{i+1}(\lambda)$ *divides* $\hat{\psi}_i(\lambda)$ *without remainder.*

$\hat{M}$ is called the *Smith-Mcmillan* form of $\hat{T}(\lambda)$.

Zeros and Poles

Theorem 21. [Zeros and poles of $\hat{T}$] *The zeros of* $\hat{T}(\lambda)$ *are the roots of* $\Pi_{i=1}^{r}\hat{\epsilon}_i(\lambda)$. *The poles of* $\hat{T}(\lambda)$ *are the roots of* $\Pi_{i=1}^{r}\hat{\psi}_i(\lambda)$.

Proof: Left to the reader. ∎

State Space Characterization of Finite Dimensional Systems

Lemma 5.

- *Let T be a FDLTIC system; $T : L^n \rightarrow L^m$. Then there exist real matrices A, B, C and D such that if $y = Tu$ for some $u \in L^n$ then*

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \\ x(0) &= 0, \end{aligned} \tag{6}$$

Proof: See C. T. Chen. ∎

The representation of the map T as given in (6) is called a *state space* representation of T. A convenient notation employed to denote the system described by (6) is $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

Controllability

Definition 27. [Controllability] *The dynamical system described by*

$$
\begin{aligned}
\dot{x} &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \qquad (7) \\
x(0) &= x_0,
\end{aligned}
$$

is said to be controllable if for any initial condition $x(0) = x_0$, $t_1 > 0$ *and final state* x_1 *there exists a piecewise continuous input* $u(.)$ *such that the solution of (7) satisfies* $x(t_1) = x_1$. *Otherwise the system or the pair* (A, B) *is said to be uncontrollable.*

Tests For Controllability

Theorem 22. *The following are equivalent*

- (A, B) *is controllable.*
- *The matrix*
$$W_c(t) := \int_0^t e^{A\tau} BB^* e^{A^*\tau} d\tau$$
is positive definite for any $t > 0$.
- *The controllability matrix*
$$C = [B \; AB \; A^2B \; \ldots \; A^{n-1}B]$$
has full row rank.

- *The matrix* $[A - \lambda I B]$ *has full row rank for all* λ *in the complex plane.*

- *Let* λ *and* x *be any eigenvalue and corresponding left eigenvector of* A *(that is* $x^* A = \lambda x^*$ *) then* $x^* B \neq 0.$

- *For any given set of* n *complex numbers* F *can be chosen such that* $A + BF$ *has the given set as its eigenvalues.*

Stabilizability

Definition 28. [Stabilzability]

The pair of real matrices A *and* B *with* $A \in R^{n \times n}$ *and with* $B \in R^{n \times m}$ *is a stabilizable pair if there exists a real matrix* K *such that* $Real(\lambda_i(A + BK)) < 0$, *where* λ_i *denotes the* i^{th} *eigenvalue.*

Tests for Stabiliability

Theorem 23. *The following are equivalent*

- (A, B) *is stabilizable.*
- *The matrix* $[A - \lambda I B]$ *has full row rank for all* λ *in the complex plane with* $Re(\lambda) \geq 0.$
- *For all* λ *and* x *that satisfy* $x^*A = \lambda x^*$ *and* $Re(\lambda) \geq 0$*,* $x^*B \neq 0.$
- F *can be chosen such that* $A + BF$ *is stable.*

Observability

Definition 29. *Observability The dynamical system described by*

$$\begin{array}{rcl} \dot{x} & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \\ x(0) & = & x_0, \end{array} \tag{8}$$

is said to be observable (or the pair (A, C) *is observable) if any initial condition* x_0 *can be determined uniquely from the output trajectory* $y(t), t \in [0, t_1]$ *where* $t_1 > 0$ *is arbitrary with* $y(t)$ *being generated by Equation (8) with* $u(\cdot) = 0$.

Test for Observability

Theorem 24. *The following are equivalent*

- (A, C) *is observable.*
- *The matrix*

$$W_o(t) := \int_0^t e^{A^*\tau} C^* C e^{A\tau} d\tau$$

  is positive definite for any $t > 0$.

- *The observability matrix*

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

  has full column rank.

- *The matrix* $\left[\begin{array}{c} A-\lambda I \\ B \end{array}\right]$ *has full column rank for all* λ *in the complex plane.*

- *Let* λ *and* y *be any eigenvalue and corresponding right eigenvector of* A *(that is* $Ax = \lambda x$ *) then* $Cx \neq 0.$

- *For any given set of* n *complex numbers* L *can be chosen such that* $A + LC$ *has the given set as its eigenvalues.*

Note that (A, C) *is observable if and only if* (A^*, C^*) *is controllable.*

Detectability

Definition 30. [Detectability] *The pair of real matrices A and C is detectable if there exists a real matrix L such that $Real(\lambda_i(A + LC)) < 0$*

Tests for Detectability

Theorem 25. *The following are equivalent*

- (A, C) *is detectable.*
- *The matrix* $\begin{bmatrix} A - \lambda I \\ B \end{bmatrix}$ *has full column rank for all* λ *in the complex plane with* $Re(\lambda) \geq 0$.
- *For all* λ *and* x *that satisfy* $Ax = \lambda x$ *and* $Re(\lambda) \geq 0$, $Cx \neq 0$.
- L *can be chosen such that* $A + LC$ *is stable.*

Note that (A, C) is detectable if and only if (A^*, C^*) is stabilizable.

Minimal Realization

Definition 31. *The triplet* (A, B, C) *is minimal if* (A, B) *is controllable and* (A, C) *is observable.*

Lemma 6. *Let* $\hat{T}$ *be a proper transfer function matrix. Then it admits a realization* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *such that* (A, B, C) *is minimal. Such a realization is called a minimal realization of* $\hat{T}$.

Continuous Time Stability Characterization

Theorem 26. *Suppose T is a FDLTIC system. Then the following statements are equivalent.*

1. *T is L_p stable for any $p, 1 \leq p \leq \infty$.*

2. *If $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is any state-space description of T such that (A, B) is stabilizable and (A, C) is detectable then $Real(\lambda_i(A)) < 0$ for all i where $\lambda_i(A)$ denotes the i^{th} eigenvalue of A.*

3. *$\hat{T}(s)$ the s-transform of T has all its poles outside the closed right half plane(that is if s_0 is a pole of $\hat{T}$ then $Real(s_0) < 0$.)*

Proof: See C. T. Chen. ■

This theorem establishes the fact that for FDLTIC systems stability in L_p sense implies stability in L_q sense for any p and q such that $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Thus for FDLTIC systems we can use the term stability to mean stability in L_p sense for any $1 \leq p \leq \infty$.

Properties of State Space Realizations

Suppose G_1 and G_2 have a state space realizations $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ and $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ respectively. Then

- $$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right].$$

- $$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

- Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is square and D is invertible then

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

Internal Stability

Definition 32. *Consider the state space dynamics described by*

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \\ x(0) & = & x_0 \end{array} .$$

The above system is said to be internally stable if for any $x_0 \in R^n$, $x(t) \to 0$ *as* $t \to \infty$ *with* $u = 0$.

Lemma 7. *The system described by the state space equations above is internally stable if and only if all the eigenvalues of the matrix* A *are in the open left half plane* $\{s \in C | Real(s) < 0\}$.

Stability of Interconnections of Multiple-input Multiple-output Systems

Interconnections

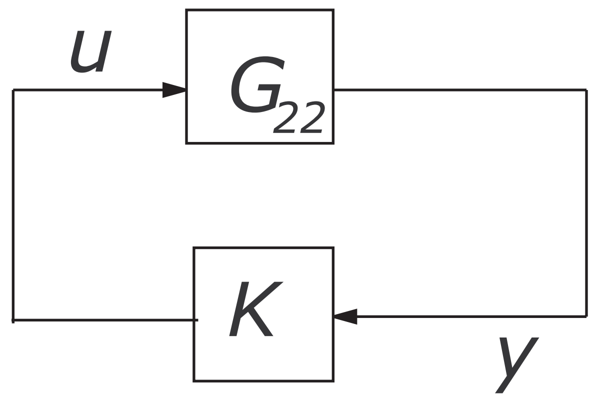


Figure 3:

Consider the interconnection represented by the block diagram in Figure 3. Suppose G_{22} and K are described by the state space descriptions

$$G_{22}\left\{\begin{array}{lcl} \dot{x}_{22}(t) & = & Ax_{22}(t)+B_2u(t) \\ y(t) & = & C_2x_{22}(t)+D_{22}u(t) \\ x(0) & = & x_0 \end{array}\right. \quad \text{and } K\left\{\begin{array}{lcl} \dot{x_K}(t) & = & A_Kx_K(t)+B_Ky(t) \\ u(t) & = & C_Kx_K(t)+D_Ky(t) \\ x_K(0) & = & x_0' \end{array}\right. \tag{9}$$

Well-posedness: State Space Viewpoint

Before analyzing the interconnection one has to address the issue of well-posedness. For the interconnection described by Figure 3 with the state space descriptions of G_{22} and K given by $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ and $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ respectively we define

Definition 33. *The interconnection described by Figure 3 governed by the (9) is well posed if there exist unique signals* $x_{22}(t), x_K(t), y(t), u(t)$ *that satisfy (9) for every initial condition* $x(0) := [x_{22}(0) \;\; x_K(0)]^T$*. Such a condition should hold for all corresponding matrices in a neighbourhood of the given matrices* $A,\ A_K,\ B_2,\ B_K,\ C_2,\ C_K,\ D_{22},$ *and* D_K.

Lemma 8. *The interconnection described by Figure 3 governed by the (9) is well posed if and only if*

$$det(I - D_{22}D_K) \neq 0.$$

Proof: ($\Rightarrow$) Assume that $(I - D_{22}D_K)$ is singular. Using (9) note that

$$\begin{array}{rcl} y & = & C_2 x_{22} + D_{22} u \\ u & = & C_K x_K + D_K y \end{array} \tag{10}$$

Thus it follows that

$$\underbrace{\begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}}_{M} \begin{pmatrix} u \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & C_K \\ C_2 & 0 \end{pmatrix}}_{N} \begin{pmatrix} x_{22} \\ x_K \end{pmatrix} \tag{11}$$

The matrix M is singular as $(I - D_{22}D_K)$ is assumed singular. Thus it is not onto (use the fact that $dim(range(M)) = rank(M)$). Let $(u_0\ y_0)$ be a vector that is not in the range space of M. It is evident that the matrix on the right hand side of the equation can be made full rank by a small perturbation if needed. Choose $(a\ b)$ such that $N(a\ b)^T = (u_0\ y_0)^T$. Thus for the initial conditions $x_{22}(0) = a$ and $x_K(0) = b$ there is no solution to the (11).

($\Leftarrow$) Assume that $det(I - D_{22}D_K) \neq 0$ and let $x_{22}(0)$ and $x_K(0)$ be given initial conditions.

Note that (9) is satisfied if and only if

$$\hat{x}_{22}(s) = (sI - A)^{-1}x_{22}(0) + (sI - A)^{-1}B_2\hat{u}(s) \quad (12)$$

$$\hat{x}_K(s) = (sI - A_K)^{-1}x_K(0) + (sI - A_K)^{-1}B_K\hat{y}(s) \quad (13)$$

$$\begin{aligned}\hat{y}(s) &= C_2\hat{x}_{22}(s) + D_{22}\hat{u}(s) \\ &= C_2[(sI - A)^{-1}x_{22}(0) + (sI - A)^{-1}B_2\hat{u}(s)] + D_{22}\hat{u}(s) \\ &= C_2(sI - A)^{-1}x_{22}(0) + [C_2(sI - A)^{-1}B_2 + D_{22}]\hat{u}(s) \\ &= \hat{e}_1(s) + G(s)\hat{u}(s) \quad (14)\end{aligned}$$

$$\hat{u}(s) = C_K\hat{x}_K(s) + D_K\hat{y}(s)$$

$$
\begin{aligned}
&= C_K[(sI - A_K)^{-1}x_K(0) + (sI - A_K)^{-1}B\hat{u}(s) + D_K\hat{y}(s)] \\
&= C_K(sI - A_K)^{-1}x_K(0) + [C_K(sI - A_K)^{-1}B_K + D_K]\hat{y}(s) \\
&= \hat{e}_2(s) + K(s)\hat{y}(s) \qquad (15)
\end{aligned}
$$

where $\hat{e}_1(s) = C_2(sI - A)^{-1}x_{22}(0)$ is determined by $x_{22}(0)$, $\hat{e}_2(s) := C_K(sI - A_K)^{-1}x_K(0)$ is determined by $x_k(0)$, $G_{22} = C_2(sI - A)^{-1}B_2 + D_{22}$ and $K = C_K(sI - A_K)^{-1}B_K + D_K$ are the transfer functions of G_{22} and K. Note that by substituting into (14) the relation (15) we have

$$
\begin{aligned}
& \hat{y}(s) &&= \hat{e}_1(s) + G_{22}(s)[\hat{e}_2(s) + K(s)\hat{y}(s)] \\
\Leftrightarrow\; & (I - G_{22}(s)K(s))\hat{y}(s) &&= \hat{e}_1(s) + G_{22}(s)\hat{e}_2(s)
\end{aligned}
$$

As $(I - D_{22}D_K)$ is invertible it follows that the transfer function $(I - G_{22}(s)K(s))^{-1}$ exists (as it has full normal rank) and is proper. Thus $y(t)$

is uniquely determined from $e_1(t)$ and $e_2(t)$ that in turn are fixed for the $x_{22}(0)$ and $x_K(0)$ given.

Note that $u(t)$ is uniquely determined causally from the equation (15) as $y(s)$ is uniquely determined.

Similarly $x_{22}(t)$ and $x_K(t)$ are uniquely determined from the equation (12) and (13) as $y(t)$ and $u(t)$ are uniquely determined. ■

Well-posedness: Input-output Viewpoint

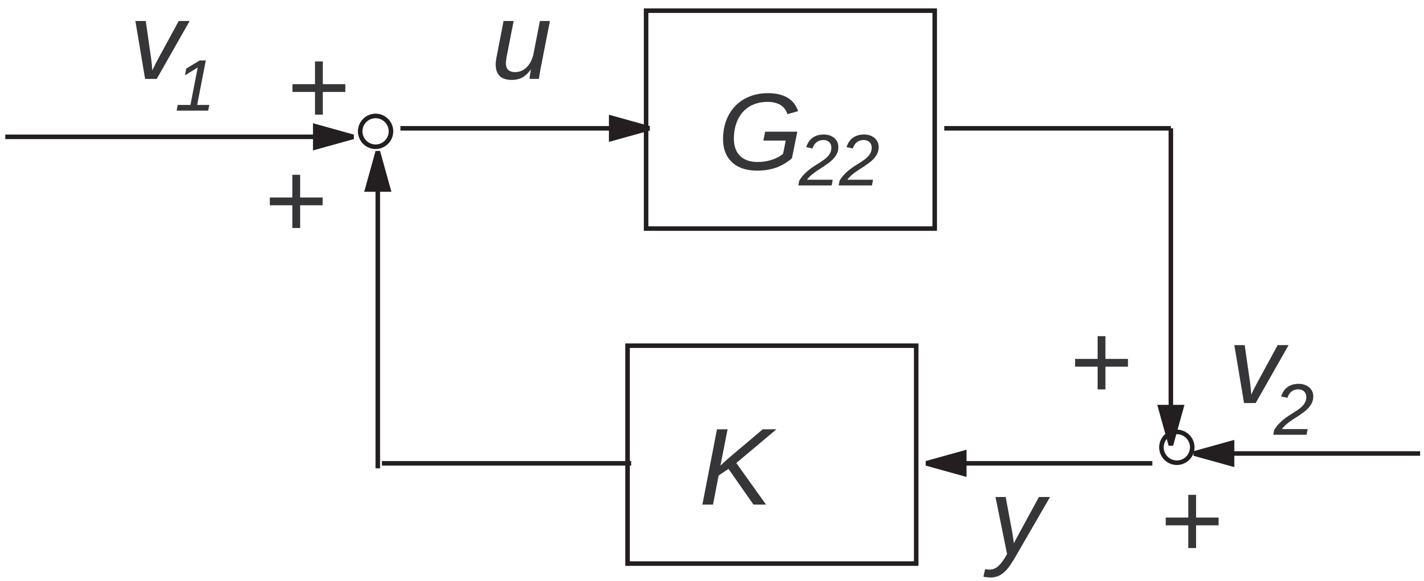


Figure 4: Parametrization of stabilizing controllers for G_{22}.

Consider the interconnection shown in Figure 4 where G_{22} and K are input-output maps whose realizations are not available. In this case the well-posedness property is defined in terms of the input signals v_1 and v_2.

Definition 34. *The interconnection shown in Figure 4 is said to be well posed if for any signals v_1 and v_2 there exist unique signals y and u satisfying the relations imposed by the interconnection and y and u should be determinable from v_1 and v_2 causally.*

Theorem 27. *The interconnection in Figure 4 is well posed if and only if*

$$I - \hat{G}_{22}(\infty)\hat{K}(\infty)$$

is invertible.

Proof: Left to the reader.

$\blacksquare$

Note that in the above interconnection it is not needed that G_{22} or K be finite dimensional.

Also, it is needed that the two well-posedness definitions should coincide when state space realizations of G_{22} and K are available.

Note that when G_{22} and K have state space realizations $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ and $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ then

$$I - \hat{G}_{22}(\infty)\hat{K}(\infty) = I - D_{22}D_K.$$

This also implies that when G_{22} and K have state space realizations then well posedness implies all transfer functions between any input-output pair is a proper transfer function. This follows from the causality requirement in well posedness.

From Theorems 27 and 8 it follows that the definitions agree.

Internal Stability of Interconnections

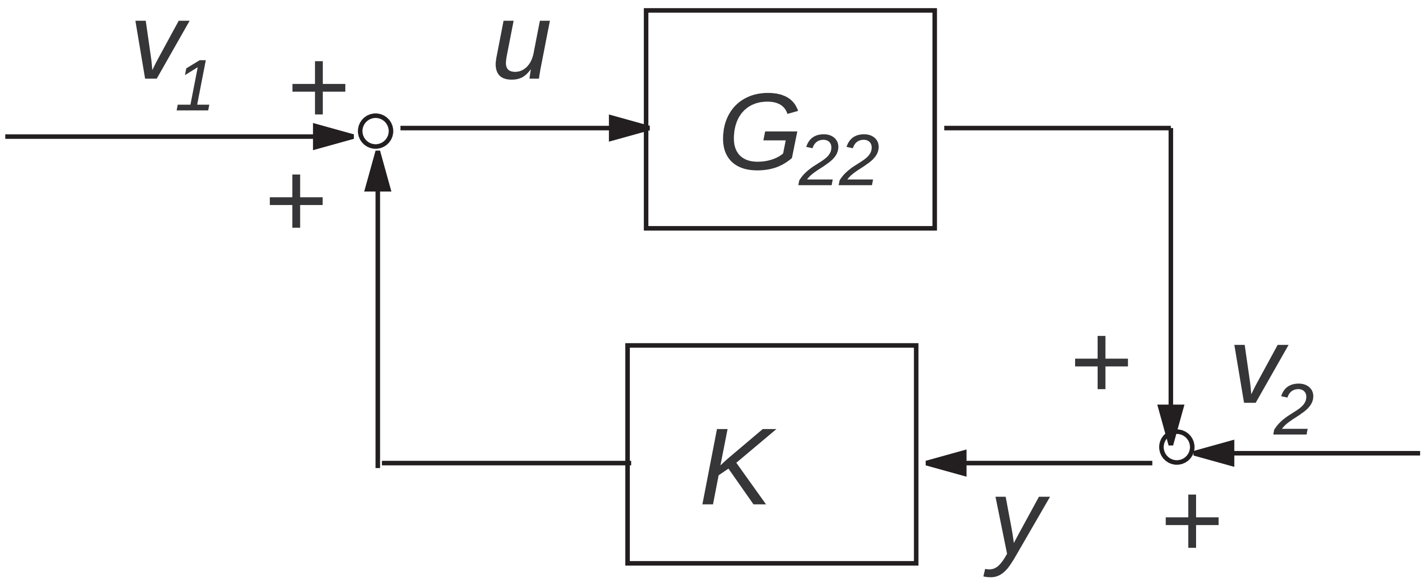


Figure 5: Parametrization of stabilizing controllers for G_{22}.

Suppose G_{22} and K have minimal state space realizations $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ and

$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$. It follows that the state space equations are described by

$$G\begin{cases} \dot{x}_{22}(t) &= Ax_{22}(t) + B_2 u(t) \\ e_1(t) &= C_2 x_{22}(t) + D_{22} u(t) \\ x(0) &= x_0 \end{cases} \quad K\begin{cases} \dot{x_K}(t) &= A_K x_k(t) + B_K y(t) \\ e_2(t) &= C_K x_k(t) + D_K y(t) \\ x_K(0) &= x_0' \end{cases} \tag{16}$$

Also

$$\begin{aligned} u &= v_1 + e_2 \\ y &= v_2 + e_1. \end{aligned}$$

Thus the interconnection is internally stable if and only if the combined state $x := (x_{22}\; x_K)^T$ converges to zero for any initial condition $x(0)$ with the inputs $v_1 = v_2 = 0$.

Lemma 9. *Let* $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ *and* $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *be minimal state space representations of* G_{22} *and* K*. Consider the input and the output of the well* *posed interconnection as*

$$w = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ and } z = \begin{pmatrix} u \\ y \end{pmatrix}$$

respectively. Let the map between w and z be denoted by $H(G_{22}, K)$. Then the state space description as given by Equation 16 is a stabilizable and detectable realization of H.

Proof:

We will first establish Controllability of the interconnection realization (16). Suppose $x(0) := [x_{22}(0) \quad x_K(0)]$ is a given initial condition and $x_d := [x_1 \quad x_2]$ is the desired final condition to be reached at time t'.

From controllability of realizations of G_{22} and K it follows that there exist signals $u(t)$ and $y(t)$ such that

$$\begin{array}{rcl} x_1 = x_{22}(t') & = & e^{At'}x_{22}(0) + \int_0^{t'} e^{A(t'-\tau)}B_2u(tau)d\tau \\ x_2 = x_K(t') & = & e^{A_Kt'}x_K(0) + \int_0^{t'} e^{A_K(t'-\tau)}B_Ky(tau)d\tau \end{array}$$

Let

$$\begin{array}{rcl} e_1 & = & C_2x_{22}(t) + D_{22}u(t) \\ e_2 & = & C_Kx_K(t) + D_Ky(t) \end{array}$$

Define the signals $v_1(t) := u(t) - e_2$ and $v_2(t) = y(t) - e_1$.

Clearly, the signals $v_1,\ v_2$ and $u,\ y$ form one input output pair that satisfy the interconnection relationship. From well posedness with v_1 and v_2 as the input $u(t)$ and $y(t)$ as defined above are the only possible signals.

Thus with v_1 and v_2 as defined it is clear that the interconnection intial condition $x(0)$ is driven to the desired state x_d.

The proof of observability is left to the reader.

■

An immediate consequence of the above lemma and Theorem 26 is the following theorem

Theorem 28. *The well posed interconnection described by Figure 5 with the minimal state space descriptions* $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ *and* $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *for* G_{22} *and* K *respectively is internally stable if and only if the map* $H(G_{22}, K)$ *is a stable map.*

Proof: Follows from Lemma 9 and Theorem 26. ■

Corollary 3. *The well posed interconnection described by Figure 5 with the minimal state space descriptions* $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ *and* $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *for* G_{22} *and* K *respectively is internally stable if and only if the maps*

$(I - G_{22}K)^{-1},\ (I - G_{22}K)^{-1}G_{22}, (I - KG_{22})^{-1},$ *and* $(I - KG_{22})^{-1}K$ *are stable.*

Proof: Note that

$$\begin{array}{rcl} u - Ky & = & v_1 \\ y - G_{22}u & = & v_2 \end{array}$$

$$\begin{aligned} \begin{pmatrix} u \\ y \end{pmatrix} &= \begin{pmatrix} I & -K \\ -G_{22} & I \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{pmatrix}}_{H(G_{22},K)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{aligned}$$

From Theorem 28 it follows that the interconnection is internally stable if all the elements of $H(G_{22}, K)$ are stable maps. ∎

This motivates the following definition of internal stability when no state space representations are available.

Definition 35. *Consider Figure 5 where G_{22} and K are linear time invariant systems (possibly infinite dimensional). Then the interconnection is internally stable in the L_2 sense if the following maps*

- $(I - G_{22}K)^{-1}$
- $(I - G_{22}K)^{-1}G_{22}$
- $(I - KG_{22})^{-1}$
- $(I - KG_{22})^{-1}K$

are in $\mathcal{H}_\infty$.

In other words (5) is internally stable if

$$\begin{pmatrix} I & -K \\ -G_{22} & I \end{pmatrix}^{-1} \in \mathcal{H}_\infty$$

Stability Theorem For MIMO Systems

Theorem 29. *Let* $n_{G_{22}}$ *and* n_K *be the number of rhp poles of* G_{22} *and* K *respectively in the interconnection shown in Figure 5. Then the interconnection is internally stable if and only if the following conditions are satisfied:*

1. *The number of rhp poles of* $L := G_{22}K$ *is equal to* $n_{G_{22}} + n_K$.
2. *The matrix transfer function* $(I - G_{22}K)^{-1}$ *is stable.*

Proof: Note that

$$\underbrace{\begin{pmatrix} I & -K \\ -G_{22} & I \end{pmatrix}}_{T} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Consider a stabilizable and detectable realization of G_{22} and K

$$G_{22}\left\{\begin{array}{lcl} \dot{x}_{22}(t) & = & Ax_{22}(t) + B_2u(t) \\ e_1(t) & = & C_2x_{22}(t) + D_{22}u(t) \\ x(0) & = & x_0 \end{array}\right. \quad K\left\{\begin{array}{lcl} \dot{x_K}(t) & = & A_Kx_K(t) + B_Ky(t) \\ e_2(t) & = & C_Kx_K(t) + D_Ky(t) \\ x_K(0) & = & x_0' \end{array}\right. \tag{17}$$

We will first obtain a stabilizable and detectable realization of T^{-1}. Note that

$$\begin{array}{lcl} v_1 & = & u - e_2 \\ v_2 & = & y - e_1 \end{array}$$

Thus a state space realization of $T := \begin{pmatrix} I & -K \\ -G_{22} & I \end{pmatrix}$ is described by

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} \begin{pmatrix} x_{22} \\ x_K \end{pmatrix} + \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 & -C_K \\ -C_2 & 0 \end{pmatrix} \begin{pmatrix} x_{22} \\ x_K \end{pmatrix} + \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}
\end{aligned}
$$

Thus T admits a state space realization

$$
T = \left[\begin{array}{c|c} \overbrace{\begin{matrix} A & 0 \\ 0 & A_K \end{matrix}}^{A} & \overbrace{\begin{matrix} B_2 & 0 \\ 0 & B_K \end{matrix}}^{B} \\ \hline \underbrace{\begin{matrix} 0 & -C_K \\ -C_2 & 0 \end{matrix}}_{-C} & \underbrace{\begin{matrix} I & -D_K \\ -D_{22} & I \end{matrix}}_{D} \end{array} \right].
$$

Thus

$$T^{-1} = \left[\begin{array}{c|c} A + BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] =: \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right]$$

where

$$\begin{aligned} \overline{D} = D^{-1} &= \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I + (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_{22} & (I - D_{22}D_K)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_{22} & (I - D_{22}D_K)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_K \\ I \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} D_{22} & I \end{pmatrix} \end{aligned}$$

Thus

$$\overline{A} = A + BD^{-1}C = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix}$$

The following are equivalent

- $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is stabilizable and detectable.
- (A, B_2, C_2, D_{22}) and (A_K, B_K, C_K, D_K) are stabilizable and detectable.

The above can be proven using the stabilizability and detectability characterization provided (see Theorem 23 and Theorem 25).

A realization of $L = G_{22}K$ is given by $\left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right]$ where

$$A_L = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix}, \; B_L = \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix}, \; C_L = (C_2 \;\; D_{22}C_K), \;\; D_L = D_{22}D_K.$$

Also

$$S = (I - L)^{-1} = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]$$

where

$$\begin{aligned} A_S &= \overline{A} = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix} \\ B_S &= \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \\ C_S &= (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix} \\ D_S &= (I - D_{22}D_K)^{-1} \end{aligned}$$

The following are equivalent:

- (A_S, B_S, C_S, D_S) is stabilizable and detectable
- (A_L, B_L, C_L, D_L) is stabilizable and detectable

The above can be proven using the stabilizability and detectability characterization provided.

Now we prove the theorem

($\Rightarrow$) Suppose the interconnection is internally stable.

This implies T^{-1} is a stable transfer function. $(\overline{A}, \overline{B}, \overline{C}, \overline{D}$ is a stabilizable and detectable realization of T^{-1} as established earlier. Thus there can be no unstable pole zero cancellations in forming the transfer matrix T^{-1}. Thus $\overline{A} = A_S$ has all eigenvalues in the open left half plane.

This implies that the realization (A_S, B_S, C_S, D_S) of S is a stabilizable and detectable realization and that S is stable.

This implies (A_L, B_L, C_L, D_L) is stabilizable and detectable (as A_S is stable there can be no unstable pole zero cancellations).

This implies (1) in the theorem statement. We have established (2) that S is stable.

($\Leftarrow$) Assume (1) and (2) to be true. Then from (1) it follows that (A_L, B_L, C_L, D_L) is stabilizable and detectable which in turn implies (A_S, B_S, C_S, D_S) is stabilizable and detectable. As S is stable, $A_S = \overline{A}$ has all eigenvalues in the open left half plane. Thus T^{-1} the interconnection matrix is stable.

This proves the theorem.

■

Nyquist Stability Criterion For MIMO Interconnections

Theorem 30. *Let $L = G_{22}K$ be such that there are no unstable pole zero cancellations while forming the product. Let the number of rhp poles of L be denoted by P_{ol}. The closed-loop interconnection of G_{22} and K is internally stable if and only if the Nyquist plot of $det(I - L(s))$*

1. *makes P_{ol} anticlockwise encirclements of the origin*

2. *does not pass through the origin.*

Proof: Condition (1) of Theorem 29 is satisfied as there are no pole-zero cancellations in forming the product $L = G_{22}K$. Condition (2) of Theorem 29 is that $S = (I - L(s))^{-1}$ be stable, that is, it should have no poles in the rhp.

As in the proof of Theorem 29 let G_{22} and K have state space stabilizable and detectable realizations $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ and $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ respectively.

A realization of $L = G_{22}K$ is given by $\left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right]$ where

$$A_L = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix}, \ B_L = \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix}, \ C_L = (C_2 \ \ D_{22}C_K), \ \ D_L = D_{22}D_K.$$

As there are no rhp pole zero cancellations in the product $G_{22}K$ it follows that above is a stabilizable and detectable realization of L. It follows that a stabilizable and detectablle realization of S is given by

$$S = (I - L)^{-1} = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]$$

where

$$
\begin{aligned}
A_S &= \overline{A} = A_L + B_L(I - D_L)^{-1}C_L \\
&= \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix} \\
B_S &= \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \\
C_S &= (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix} \\
D_S &= (I - D_{22}D_K)^{-1}
\end{aligned}
$$

Thus the S is a stable map if and only if all eigenvalues of the A_S are in the lhp. Thus the stability of S is characterized by the zeros of the polynomial

$$\phi_{cl}(s) = det(sI - A_L - B_L(I - D_L)^{-1}C_L).$$

We will use the following result which is called the Schur's formula:

$$
\begin{aligned}
det\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} &= det(P_{11})det(P_{22} - P_{21}P_{11}^{-1}P_{12}) \\
&= det(P_{22})det(P_{11} - P_{12}P_{22}^{-1}P_{21})
\end{aligned}
$$

.

From above it follows that

$$
\begin{aligned}
\phi_{cl}(s)det(I - D_L) &= det(\underbrace{sI - A_L}_{P_{22}} - B_L(\underbrace{I - D_L}_{P_{11}})^{-1}C_L)det(I - D_L) \\
&= det(sI - A_L)det[(I - D_L) - C_L(sI - A_L)^{-1}B_L] \\
&= \phi_{ol}(s)det(I - L(s))
\end{aligned}
$$

This implies that

$$
det(I - L(s)) = \frac{\phi_{cl}(s)}{\phi_{ol}(s)}c
$$

where c is a constant and ϕ_{cl} and $\phi_{ol}(s)$ are polynomials in s. Let N be the number of clockwise encirclements of the origin of $det(I - L(s))$ as s varies over the Nyquist contour. Let P_{ol} and Z be the number of zeros in the rhp of $\phi_{ol}(s)$ and $\phi_{cl}(s)$. From the Argument principle we have

$$N = Z - P_{ol}.$$

Stability of S is guaranteed if and only if $Z = 0$. Thus S is stable if and only if the Nyquist plot of $det(I - L(s))$ should encircle the origin P_{ol} times in the counterclockwise direction without touching the origin.

This proves the theorem.

$\blacksquare$.

Parametrization of Stabilizing Controllers

Coprime Factorization for MIMO Systems

Definition 36. [rcf, lcf, dcf] *Stable FDLTIC systems M and N are right coprime if there exist stable FDLTIC systems X and Y such that the s-transforms satisfy the identity*

$$\tilde{X}(s)M(s) - \tilde{Y}(s)N(s) = I. \tag{18}$$

Stable FDLTIC systems $\tilde{M}$ and $\tilde{N}$ are left coprime if there exist stable FDLTIC systems $\tilde{X}$ and $\tilde{Y}$ such that the s-transforms satisfy the identity

$$\tilde{M}(s)X(s) - \tilde{N}(s)Y(s) = I. \tag{19}$$

Suppose $T = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ where N and M are right coprime and $\tilde{M}$ and $\tilde{N}$ are left coprime. Then the pair N and M form a right coprime factorization (rcf) of T and the pair $\tilde{M}$ and $\tilde{N}$ form a left coprime factorization (lcf) of T.

A doubly-coprime factorization (dcf) of a FDLTIC system T is a set of stable

FDLTIC maps $M, N, \tilde{M}$ *and* $\tilde{N}$ *such that* $T = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ *and*

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I. \tag{20}$$

Note that the dcf identity is a compact way of expressing

$$\begin{aligned} \tilde{X}(s)M(s) - \tilde{Y}(s)N(s) &= I \\ \tilde{M}(s)X(s) - \tilde{N}(s)Y(s) &= I \\ NM^{-1} &= \tilde{M}^{-1}\tilde{N} \\ YX^{-1} &= \tilde{X}^{-1}\tilde{Y} \end{aligned}$$

Interconnection of FDLTIC Systems

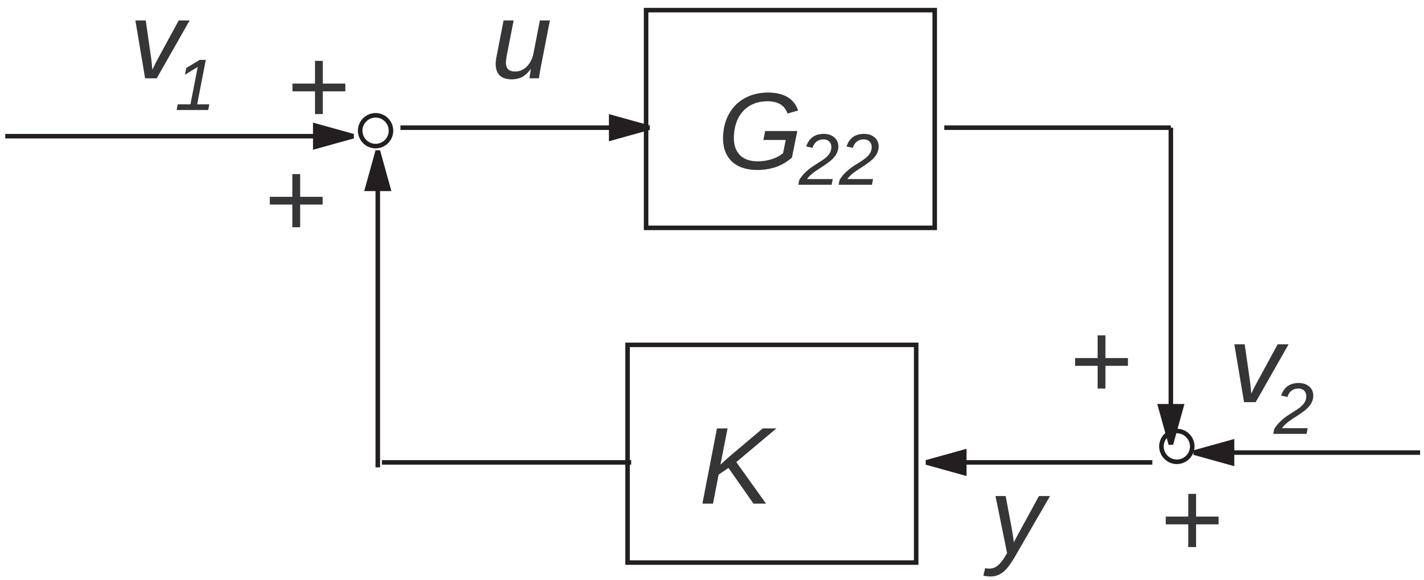


Figure 6: Parametrization of stabilizing controllers for G_{22}.

Lemma 10. *Let G_{22} be a FDLTIC system which has a dcf given by*

$G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ *where*

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I. \tag{21}$$

A FDLTIC controller K stabilizes the closed loop map shown in Figure 6 if and only if K has a rcf $K = Y_1 X_1^{-1}$ such that the map

$$\begin{pmatrix} M & Y_1 \\ N & X_1 \end{pmatrix}$$

has a stable inverse.

Proof:

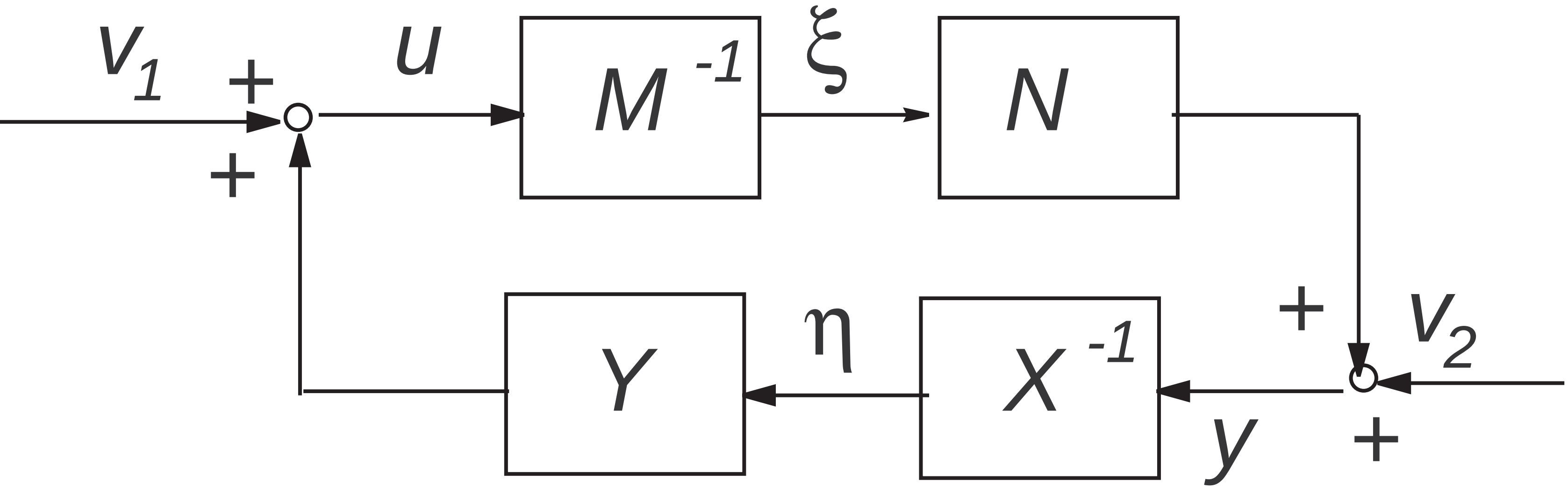


Figure 7:

($\Leftarrow$) Suppose an rcf of K is given by $Y_1 X_1^{-1}$ and suppose $\begin{pmatrix} M & Y_1 \\ N & X_1 \end{pmatrix}^{-1}$ is stable. It is clear that the Figure 6 is the equivalent to Figure 7. Note that the map from $(\xi,\ \eta)$ to $(v_1,\ v_2)$ is given by $\begin{pmatrix} M & -Y_1 \\ -N & X_1 \end{pmatrix}$. Because the inverse of this map is stable it follows that the map from $(v_1,\ v_2)$ to $(\xi,\ \eta)$ is stable. But $\|y\|_p = \|N\xi + v_2\|_p \leq \|N\|_{p-ind}\|\xi\|_p + \|v_2\|_p$ and $\|u\|_p = \|Y_1\eta + v_1\|_p \leq \|Y_1\|_{p-ind}\|\eta\|_p + \|v_1\|_p$. Thus the map from $(v_1,\ v_2)$ to $(u,\ y)$ is stable and therefore the closed loop map is stable.

($\Rightarrow$)Let FDLTIC controller K be such that the closed loop map in Figure 6 is stable. Thus the map from $(v_1,\ v_2)$ to $(u,\ y)$ is stable. Every FDLTIC system admits a dcf (see Lemma 13), and therefore it admits a rcf also. Let a rcf of K be given by $K = Y_1X_1^{-1}$. From the dcf of G_{22} it is follows that $\tilde{X}M - \tilde{Y}N = I$. Multiplying both sides of this equation by ξ we have $\xi = \tilde{X}(u) - \tilde{Y}(y - v_2)$ and thus $\|\xi\|_p \leq \|\tilde{X}\|_{p-ind}\|u\|_p + \|\tilde{Y}\|_{p-ind}(\|y\|_p + \|v_2\|_p)$. This implies that the map from $(v_1,\ v_2)$ to (ξ, η) is stable. Thus the inverse of the map $\begin{pmatrix} M & -Y_1 \\ -N & X_1 \end{pmatrix}$ is stable. ■

Youla Parametrization

Theorem 31. *Let FDLTIC system G_{22} admit a dcf as given in Lemma 10. Then K is a FDLTIC stabilizing controller for the closed loop system in Figure 6 if and only if*

$$K = (Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}),$$

for some FDLTIC stable system Q.

Proof:($\Leftarrow$) Multiplying both sides of (21) by $\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}$ from the left and by $\begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix}$ from the right we have

$$\begin{pmatrix} \tilde{X} - Q\tilde{N} & -\tilde{Y} + Q\tilde{M} \\ -\tilde{N} & -\tilde{M} \end{pmatrix} \begin{pmatrix} M & Y - MQ \\ N & X - NQ \end{pmatrix} = I, \qquad (22)$$

where Q is a stable FDLTIC map. From Lemma 10 it follows that $K = (Y - MQ)(X - NQ)^{-1}$ is a stabilizing controller. It also follows from (22) that $(Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})$ (follows from the observation that the $(1, 2)$ element of the product in (22) is zero).

($\Rightarrow$) Suppose K is a stabilizing controller. Then from Lemma 10 we know that there exist stable FDLTIC systems Y_1 and X_1 such that $K = Y_1 X_1^{-1}$ and $\begin{pmatrix} M & Y_1 \\ N & X_1 \end{pmatrix}^{-1}$ is stable. Thus it folows that

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y_1 \\ N & X_1 \end{pmatrix} = \begin{pmatrix} I & -QD \\ 0 & D \end{pmatrix}$$

is stable with a stable inverse where $D = -\tilde{N}Y_1 + \tilde{M}X_1$ and $Q := -(\tilde{X}Y_1 - \tilde{Y}X_1)D^{-1}$. Therefore D is stable with a stable inverse. Thus D^{-1} is a stable system and therefore Q is also stable. Multiplyng both sides of the above equation by $\begin{pmatrix} M & Y \\ N & X \end{pmatrix}$ we have

$$\begin{pmatrix} M & Y_1 \\ N & X_1 \end{pmatrix} = \begin{pmatrix} M & (Y - MQ)D \\ N & (X - NQ)D \end{pmatrix}.$$

By comparing entries in the above equality we have the result that $K = (Y - MQ)(X - NQ)^{-1}$. This proves the theorem. $\blacksquare$.

Generalized Plant

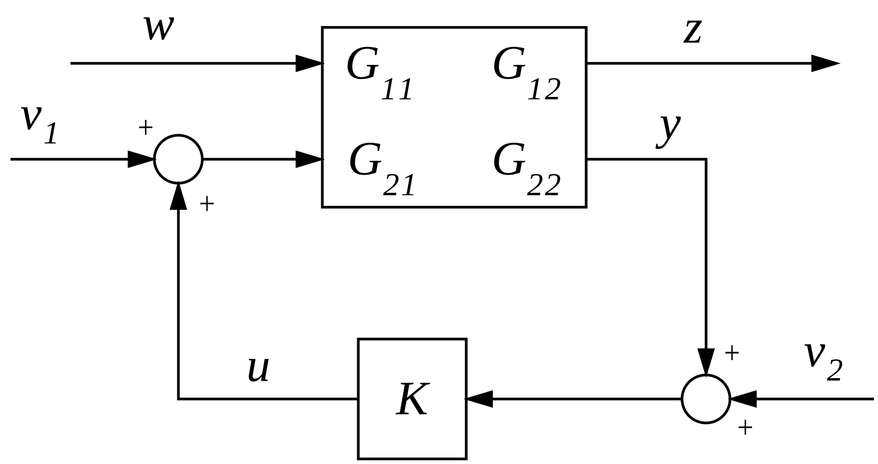


Figure 8:

Many control design issues can be cast into the framework shown in Figure 8.

- G is the generalized plant.
- w is the exogenous input

- u is the control inputs
- y is measured output
- z is the regulated output.
- K is the controller which maps the measured outputs y to control inputs u when v_1 and v_2 are zero.

Both K and G are linear systems. With respect to the interconnection of systems G and K in Figure 8, the first issue that needs to addressed is the existence and uiqueness of signals z, u and y for given input signals w, v_1 and v_2.

Well-posedness of Interconnection

Definition 37. *The interconnection in Figure 8 is well posed if for arbitrary inputs w, v_1 and v_2, u and y can be uniquely determined from w, v_1 and v_2 in a causal manner.*

An equivalent definition in the case $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ are FDLTIC is

Definition 38. *The interconnection in Figure 8 is well posed if for arbitrary initial conditions $x_G(0)$ and $x_K(0)$ the dynamics*

$$\begin{array}{rcl} \dot{x}_G(t) & = & Ax_G(t) + B\begin{pmatrix} w \\ u(t) \end{pmatrix} \\ y(t) & = & Cx_{22}(t) + D\begin{pmatrix} w \\ u(t) \end{pmatrix} \end{array} \quad \textit{and} \quad \begin{array}{rcl} \dot{x_K}(t) & = & A_K x_K(t) + B_K y(t) \\ u(t) & = & C_K x_K(t) + D_K y(t) \end{array} \tag{23}$$

where $w = v_1 = v_2 = 0$ *has trajectories* $x_G(t)$ *and* $x_K(t)$ *uniquely defined that satisfy (23). The above condition must hold for arbitrarily small perturbations of the state space matrices.*

Well Posedness

Let us assume that in Figure 8 G and K are FDLTIC systems. Also assume that a stabilizable and detectable state-space description of G is described by

$$G = \begin{pmatrix} G_{11} & G_{22} \\ G_{21} & G_{22} \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{22} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

This notation is a convenient way of writing

$$G_{11} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right],\ G_{12} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right],\ G_{21} = \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right]$$

and $G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right].$

Theorem 32. *With the state space representations of G and K as given* *above, the interconnection is well posed if and only if*

$$det(I - D_{22}D_K) \neq 0.$$

Proof:: Left to the reader (Follows similar arguments as provided in the proof of well posedness of G_{22} and K interconnection. ■

Input Output Map

Note that for the interconnection in Figure 8 the existence and uniqueness of z, u and y is sufficient for the well-posedness of the interconnection. The signals satisfy the relation

$$\begin{pmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{pmatrix} \begin{pmatrix} z \\ u \\ y \end{pmatrix} = \begin{pmatrix} G_{11} & 0 & 0 \\ 0 & I & K \\ G_{21} & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix}. \tag{24}$$

We will suppose throughout that the interconnection is well-posed. This is guaranteed if the map G_{22} is strictly causal. Let $H(G, K)$ be such that

$$\begin{pmatrix} z \\ u \\ y \end{pmatrix} = H(G, K) \begin{pmatrix} w \\ v_1 \\ v_2 \end{pmatrix}.$$

The interconnection described by $H(G, K)$ is often referred to as the *closed* *loop map*.

Stability of Closed Loop Maps

Definition 39. [Stability of closed loop maps] *The closed loop map described by Figure 8 is ℓ_p stable if $\|H(G,K)\|_{p-ind} < \infty$. In such a case K is said to be a stabilizing controller in the ℓ_p sense.*

Lemma 11. *There exists a FDLTIC system K which stabilizes the closed loop in Figure 8 if and only if (A, B_2) is stabilizable and (A, C_2) is detectable. If F and L are such that $A + B_2F$ and $A + LC_2$ are stable matrices then a controller with a state space realization given by*

$$K = \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right], \tag{25}$$

stabilizes the closed loop system depicted in Figure 8.

Proof:($\Leftarrow$) If (A, B_2) is stabilizable and (A, C_2) is detectable then there exist matrices F and L are such that $A + B_2F$ and $A + LC_2$ are stable. Let K be a controller with a state space realization given in (25). It can be shown that the closed loop system has a state-space description given by $\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ where

$$\tilde{A} = \begin{pmatrix} A & B_2F \\ -LC_2 & A + B_2F + LC_2 \end{pmatrix},$$

which has the same eigenvalues as the matrix

$$\begin{pmatrix} A + LC_2 & 0 \\ -LC_2 & A + B_2F \end{pmatrix}.$$

Thus $\tilde{A}$ is stable from which it follows from Theorem 26 that the closed loop map is stable.

($\Rightarrow$) If (A, B_2) is not stabilizable or (A, C_2) is not detectable then some eigenvalues of $\tilde{A}$ will remain unstable for any FDLTIC controller K. Details are left to the reader. ■

The controller K given above is called the **Luenberger** controller

Lemma 12. *Suppose (A, B_2) is stabilizable and (A, C_2) is detectable. Then FDLTIC system K stabilizes the closed loop system depicted in Figure 8 if and only if it stabilizes the closed loop system depicted in Figure 6.*

Proof:($\Rightarrow$) The closed loop map depicted in Figure 8 is described by the equations

$$\begin{aligned} z &= G_{11}w + G_{12}u \\ y &= G_{21}w + G_{22}u \\ u &= Ky + Kv_2 + v_1. \end{aligned} \tag{26}$$

The description of the closed loop map depicted in Figure 6 is given by

$$\begin{aligned} y &= G_{22}u \\ u &= Ky + Kv_2 + v_1. \end{aligned} \tag{27}$$

It is thus clear (substitute $w = 0$ in (26)) that if the map from $(w,\ v_1,\ v_2)$ to $(z,\ u,\ y)$ in (26) is stable then map from $(v_1,\ v_2)$ to $(u,\ y)$ in (27) is stable.

($\Leftarrow$) Suppose K is a stabilizing controller for the closed loop map in Figure 6. Let $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ be a stabilizable and detectable state-space description of K. By assumption $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ is a stabilizable and detectable state-space description of G_{22}. Suppose, $\left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right]$ is a state-space description of the closed loop map obtained by employing the aforementioned state-space descriptions of G_{22} and K. Then one can show that $(\overline{A}, \overline{B})$ and $(\overline{A}, \overline{C})$ are stabilizable and detectable. Thus from Theorem 26 it follows that $\overline{A}$ is stable.

If $\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$ is a description of the closed loop map in Figure 8 obtained by using the descriptions $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ for K and $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for G_{22} then by computing $\tilde{A}$ one can verify that $\tilde{A} = \overline{A}$. Thus $\tilde{A}$ is stable and therefore from Theorem 26 it follows that the closed loop system in Figure 8 is stable. $\blacksquare$

Youla Parametrization of Stabilizing Controllers

Theorem 33. *Suppose (A, B_2) is stabilizable and (A, C_2) is detectable. Let FDLTIC system G_{22} admit a dcf as given in Lemma 10. Then K is a FDLTIC stabilizing controller for the closed loop system in Figure 8 if and only if*

$$K = (Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}),$$

for some FDLTIC stable system Q.

Proof: Follows immediately from Theorem 31 and Lemma 12. ∎

Youla Parametrization of Closed-loop Maps

- By using the above parametrization we can show that

$$K(I - G_{22}K)^{-1} = (Y - MQ)\tilde{M}.$$

The map from w to z in Figure 8 is given by

$$\Phi = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

Thus we have the folowing theorem

Theorem 34. *Let G be FDLTIC system and let G_{22} admit a dcf as given in Lemma 10. Φ is a map from w to z in Figure 8 for some FDLTIC, K which stabilizes the closed loop if and only if*

$$\Phi = H - UQV,$$

where

$$\begin{aligned} H &= G_{11} + G_{12} Y \tilde{M} G_{21} \\ U &= G_{12} M \\ V &= \tilde{M} G_{21} \end{aligned}$$

and Q is some stable FDLTIC system.

We now present a result which is a generalization of Theorem 34.

Theorem 35. [Youla parametrization] *Let G be a FDLTIC system and let G_{22} admit a dcf as given in Lemma 10. Φ is a map from w to z in Figure 8 for some linear, time invariant, causal K which stabilizes the closed loop in the ℓ_∞ sense if and only if*

$$\Phi = H - UQV,$$

where

$$\begin{aligned} H &= G_{11} + G_{12} Y \tilde{M} G_{21} \\ U &= G_{12} M \\ V &= \tilde{M} G_{21} \end{aligned}$$

and Q is some ℓ_∞ stable system.

The parameter Q is often referred to as the Youla parameter. The difference between Theorem 34 and Theorem 35 is that in Theorem 35 the controller K is not restricted to be finite-dimensional. The proof of this theorem is similar to the one presented for Theorem 34 except that an analogous result for coprime factorization over ℓ_∞ stable systems is utilized.

Existence of Coprime Factors

Lemma 13. *Let T be a FDLTIC map with a state space description $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Suoppse$(A, B)$ is stabilizable and (A, C) is detectable. Then there exists a dcf of T.*

Proof: In the definition of dcf let

$$\begin{aligned}
X &= \left[\begin{array}{c|c} A+BF & -L \\ \hline C+DF & I \end{array}\right], & Y &= \left[\begin{array}{c|c} A+BF & -L \\ \hline F & 0 \end{array}\right], \\
\tilde{X} &= \left[\begin{array}{c|c} A+LC & B+LD \\ \hline F & -I \end{array}\right], & \tilde{Y} &= \left[\begin{array}{c|c} A+LC & L \\ \hline F & 0 \end{array}\right], \\
M &= \left[\begin{array}{c|c} A+BF & -B \\ \hline F & -I \end{array}\right], & N &= \left[\begin{array}{c|c} A+BF & -B \\ \hline C+DF & -D \end{array}\right], \\
\tilde{M} &= \left[\begin{array}{c|c} A+LC & -L \\ \hline -C & I \end{array}\right], \text{ and} & \tilde{N} &= \left[\begin{array}{c|c} A+LC & B+LD \\ \hline C & D \end{array}\right].
\end{aligned}$$

Then it can be shown that $T = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and (20) is satisfied. ■

Multiple Input Multiple Output Interconnections: Performance

Definition 40. [Group] *A group is a set G with a binary operation $(.) : G \times G \to G$ defined which has the following properties.*

1. $(a.b).c = a.(b.c)$; *associativity property.*

2. *There exists an element e in G such that $a.e = e.a = a$ for all a in G. e is called the identity.*

3. *For every a in G there exists an element a^{-1} in G such that $a.a^{-1} = a^{-1}.a = e$. a^{-1} is called the inverse of a.*

Definition 41. [Subgroup] *If H is a subset of a group G the H is a subgroup if H is a group with the binary operation inherited from G.*

Lemma 14. *H is a subgroup of the group G if the identity element e is in H, a belongs to H implies a^{-1} is in H and a and b belong to H implies $a.b$ belongs to H.*

Lemma 15. *A group G has a unique identity element. Also, every element in G has a unique inverse.*

Definition 42. [Abelian group] *A group G is an abelian group if for any two elements in G, $a.b = b.a$.*

Definition 43. [Homomorphism] *Let G and H be two groups. $\phi : G \rightarrow H$ is a homomorphism between the two groups if $\phi(a.b) = \phi(a).\phi(b)$, for all a, b in G.*

Lemma 16. *A homomorphism $\phi : G \to H$ sends identity of G to the identity of H and sends inverses to inverses.*

Definition 44. [Isomorphism] *An isomorphism is a homomorphism which is one to one and onto.*

Definition 45. [Fields] *A field K is a set that has the operations of addition $(+) : K \times K \to K$ and multiplication $(.) : K \times K \to K$ defined such that*

1. *multiplication distributes over addition*

$$a.(b + c) = a.b + a.c,$$

2. *K is an abelian group under addition with identity written as 0 for addition.*

3. *$K \backslash \{0\}$ is an abelian group under multiplication with identity being 1.*

Lemma 17. *If in a field K elements $a \neq 0$ and $b \neq 0$ then $ab \neq 0$.*

Vector Space

Definition 46. *A set V with two operations addition $(+) : V \times V \rightarrow V$ and scalar multiplication $(.) : V \times K \rightarrow V$, where K is a field defined is a vector space over the field K if*

1. *V is an abelian group under addition.*
2. *multiplication distributes over addition*

$$\alpha.(b + c) = \alpha.a + \alpha.b, \text{ for all } \alpha \text{ in } K, \text{ for all } a, b \text{ in } V.$$

The elements of the field K are often called as scalars. The vector space is called a real vector space if the field $K = R$ and the vector space is called a complex vector space if the field $K = C$.

Definition 47. [Algebra] *V a vector space is an algebra if it has an operation vector multiplication $(\cdot) : V \times V \to V$ defined such that this operation distributes over vector addition.*

Definition 48. [Units] *If A is an algebra then x in A is an unit if there exists some y in A such that $x \cdot y = y \cdot x = 1$.*

Lemma 18. *If A is an algebra with an associative vector multiplication and U is the set of units in A then U is a group under vector multiplication.*

From now on we will restrict the field to be either the set of real numbers R or the set of complex numbers C. Thus when we say K we mean either R or C.

Normed Vector Space

Definition 49. *A normed linear space is a vector space X with a function $||\cdot|| : X \to R$ defined such that*

1. $||x|| \geq 0$ *and* $||x|| = 0$ *if and only if* $x = 0$.
2. $||\alpha x|| = |\alpha|\ ||x||$ *for any scalar* α *and vector* x *in* X.
3. $||x + y|| \leq ||x|| + ||y||$.

Definition 50. [Induced Norm] *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let $A : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a map. The induced norm of the operator A is defined by*

$$\|A\|_{ind} = \sup_{x \neq 0} \frac{\|A(x)\|_Y}{\|x\|_X}.$$

Example 12. *Let A be a $m \times n$ matrix. Thus $A : R^n \to R^m$. Let the norm on R^m and R^n spaces be the ∞ norm ($\|x\|_\infty = \max_i |x_i|$ where $x = (x_1, \ldots, x_k)^T$.) Then the infinity induced norm is given by*

$$\|A\|_{\infty - ind} = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|.$$

Proof: Note that

$$\begin{array}{rcl} \|Ax\|_\infty & = & \max_i |\sum_{j=1}^n a_{ij} x_j| \\ & \leq & \max_i \sum_{j=1}^n |a_{ij}| \; |x_j| \\ & \leq & \|x\|_\infty \max_i \sum_{j=1}^n |a_{ij}| \end{array}$$

Thus

$$\max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_i \sum_{j=1}^{n} |a_{ij}|.$$

Suppose

$$i_0 = arg\{\max_i \sum_{j=1}^{n} |a_{ij}|\}.$$

Let $\bar{x}_j = sgn(a_{ij})$. Then it follows that

$$\begin{array}{rcl} \frac{\|A\bar{x}\|_\infty}{\|\bar{x}\|_\infty} & = & \max_i |\sum_{j=1}^n a_{ij}\bar{x}_j| \\ & \geq & |\sum_{j=1}^n a_{i_0 j}\bar{x}_j| \\ & = & \sum_{j=1}^n |a_{i_0 j}| \\ & = & \max_i \sum_{j=1}^n |a_{ij}| \end{array}$$

Therefore

$$\max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \geq \max_i \sum_{j=1}^{n} |a_{ij}|.$$

$\blacksquare$

Lemma 19. *Let $x \in \mathcal{C}^n$ and $y \in \mathcal{C}^m$.*

1. *Suppose $n \geq m$. Then $\|x\|_2 = \|y\|_2$ if and only if there exists a matrix $U \in \mathcal{C}^{n \times m}$ such that $x = Uy$ and $U^*U = I$.*

2. *Suppose $n = m$. Then $|x^*y| \leq \|x\|_2 \|y\|_2$. Moreover the equality holds if and only if $x = \alpha y$ for some $\alpha \in \mathcal{C}$ or $y = 0$.*

3. *$\|x\| = \|y\|$ if and only if there is a matrix $\Delta \in \mathcal{C}^{n \times m}$ with $\|\Delta\|_{2-ind} \leq 1$ such that $x = \Delta y$. Furthermore $\|x\| < \|y\|$ if and only if $\|\Delta\|_{2-ind} < 1$.*

4. *$\|Ux\|_2 = \|x\|_2$ for any unitary matrix U.*

Lemma 20. *Let A and B be matrices with appropriate dimensions. Then*

1. $\rho(A) \leq \|A\|$ *where $\|\cdot\|$ is any induced norm.*

2. $\|AB\| \leq \|A\| \, \|B\|$ *where $\|\cdot\|$ denotes any induced norm.*

3. $\|UAV\|_{2-ind} = \|A\|_{2-ind}$ *where U and V are unitary matrices.*

Theorem 36. *Let $A \in \mathcal{C}^{m\times n}$. Then there exists unitary matrices $U \in \mathcal{C}^{m\times m}$ and $V \in \mathcal{C}^{n\times n}$ such that*

$$A = U\Sigma V^*$$

such that

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with $\Sigma_1 = diag(\sigma_1, \sigma_2, \ldots, \sigma_p)$ where $p = \min\{m, n\}$ and $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_p \geq 0$. σ_i are called the singular values of A.

Example 13. *Let $A \in \mathcal{C}^{m\times n}$. Thus $A : \mathcal{C}^n \rightarrow \mathcal{C}^m$. The two induced norm of A is its maximum singular value.*

Proof: Note that

$$\begin{aligned}
\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} &= \max_{x \neq 0} \frac{\|U\Sigma V^* x\|_2}{\|x\|_2} \\
&= \max_{x \neq 0} \frac{\sqrt{(U\Sigma V^* x)^* (U\Sigma V^* x)}}{\|x\|_2} \\
&= \max_{x \neq 0} \frac{\sqrt{x^* V \Sigma^* U^* U \Sigma V^* x}}{\|x\|_2} \\
&= \max_{x \neq 0} \frac{\sqrt{x^* V \Sigma^* \Sigma V^* x}}{\|V^* x\|_2} \\
&= \max_{x \neq 0, y = V^* x} \frac{\sqrt{y^* \Sigma^* \Sigma y}}{\|y\|_2} \\
&= \max_{x \neq 0, y = V^* x} \frac{\sqrt{\sigma_1^2 |y_1|^2 + \sigma_2^2 |y_2|^2 + \ldots + \sigma_p^2 |y_p|^2}}{\|y\|_2} \\
&\leq \max_{x \neq 0, y = V^* x} \frac{\sqrt{\sigma_1^2 |y_1|^2 + \sigma_1^2 |y_2|^2 + \ldots + \sigma_1^2 |y_p|^2}}{\|y\|_2} \\
&\leq \sigma_1 \max_{x \neq 0, y = V^* x} \frac{\sqrt{|y_1|^2 + |y_2|^2 + \ldots + |y_p|^2}}{\|y\|_2} \\
&\leq \sigma_1.
\end{aligned}$$

Let

$$\bar{x} = Ve_1$$

where $e_1 = (1, 0, 0 \ldots, 0)^T$. Then it follows that

$$\begin{aligned} \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} &\geq \frac{\|A\bar{x}\|_2}{\|\bar{x}\|_2} \\ &= \|U\Sigma V^* \bar{x}\|_2 \\ &= \|U\Sigma V^* V e_1\|_2 \\ &= \|U\Sigma e_1\|_2 \\ &= \sigma_1 \end{aligned} .$$

Therefore

$$\|A\|_{2-ind} = \sigma_1.$$

■

Also the notation $\bar{\sigma}(A) = \sigma_1$ is used to denote the maximum singular value of A and $\underline{\sigma}(A) = \sigma_p$ is utilized to denote the smallest singular value of A.

It follows from $U\Sigma V^* = A$ that

$$\begin{aligned} Av_i &= \sigma_i u_i \\ A^* u_i &= \sigma_i v_i \end{aligned}$$

Thus

$$\begin{aligned} A^* A v_i &= \sigma_i A^* u_i = \sigma_i^2 v_i \\ A A^* u_i &= \sigma_i A v_i = \sigma_i^2 u_i \end{aligned}$$

Thus σ_i^2 are the eigenvalues of A^*A and AA^*.

Lemma 21. *1.* $\bar{\sigma}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.

2. $\underline{\sigma} = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.

3. $|\underline{\sigma}(A + \Delta) - \underline{\sigma}(A)| \leq \bar{\sigma}(\Delta)$.

4. $\underline{\sigma}(A\Delta) \geq \underline{\sigma}(A)\underline{\sigma}(\Delta)$

5. $\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$.

The $\mathcal{L}_\infty$ and the $\mathcal{H}_\infty$ spaces

- $\mathcal{L}_\infty(jR)$ is the Banach space of matrix valued functions that are essentially bounded on the imaginary axis with the norm

$$\|F\|_\infty := ess \sup_{\omega \in R} \bar{\sigma}[F(j\omega)].$$

- $\mathcal{H}_\infty$ is the Banach space of matrix valued functions that are essentially bounded on the imaginary axis and analytic in the closed right half plane $\{s : Re(s) \geq 0\}$ with the norm

$$\|F\|_\infty := \sup_{s:Re(s)\geq 0} \bar{\sigma}[F(s)] = ess \sup_{\omega \in R} \bar{\sigma}[F(j\omega)].$$

- $\mathcal{RH}_\infty$ is the subspace of $\mathcal{H}_\infty$ that consist of elements that are real and rational functions of the complex variable s.

The $\mathcal{H}_\infty$ norm is the two induced norm

Suppose $G \in \mathcal{L}_\infty$ be a $p \times q$ transfer matrix. Consider the multiplication operator induced by G on

$$\mathcal{L}_2 = \{f : \|f\|_2 := \int_{-\infty}^{\infty} |f(j\omega)|^2 d\omega < \infty\|$$

defined by

$$M_G : \mathcal{L}_2 \to \mathcal{H}_2; M_G f = Gf.$$

Theorem 37. *Let $G \in \mathcal{L}_\infty$ be a $p \times q$ transfer matrix. Then*

$$\|M_G\| := \|M_G\|_{2-ind} = \|G\|_\infty.$$

Proof: Note that

$$\|M_G\| = \sup\{\|Gf\|_2 : \|f\|_2 \le 1\}.$$

Now

$$\begin{aligned}
\|Gf\|_2^2 &= \int_{\infty}^{\infty} f^*(j\omega)G^*(j\omega)G(j\omega)F(j\omega)d\omega \\
&= \int_{\infty}^{\infty} \|G(j\omega)f(j\omega)\|_2^2 \\
&\leq \int_{\infty}^{\infty} \bar{\sigma}^2[G(j\omega)]\|f(j\omega)\|_2^2 \\
&\leq \|G\|_\infty \int_{\infty}^{\infty} \|f(j\omega)\|_2^2
\end{aligned}$$

Thus

$$\|M_G\| = \sup\{\|Gf\|_2 : \|f\|_2 \leq 1\} \leq \|G\|_\infty.$$

This proves one side of the theorem. That $\|M_G\| \geq \|G\|_\infty$ is left as an exercise.

$\blacksquare$

Consider a $p \times q$ MIMO transfer matrix G. Let $y = G(s)u$ with

$$y(t) = \begin{pmatrix} y_1 \sin(\omega_0 t + \phi_1) \\ y_2 \sin(\omega_0 t + \phi_2) \\ \vdots \\ y_p \sin(\omega_0 t + \phi_p) \end{pmatrix} \text{ and } u(t) = \begin{pmatrix} u_1 \sin(\omega_0 t + \theta_1) \\ u_2 \sin(\omega_0 t + \theta_2) \\ \vdots \\ u_q \sin(\omega_0 t + \theta_q) \end{pmatrix}$$

It can be shown that

$$\sup_{\theta_i, \omega_0, \|\bar{u}\|_2} \|\bar{y}\|_2 = \|G\|_\infty$$

where

$$\bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \text{ and } \bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{pmatrix}$$

Performance Specifications

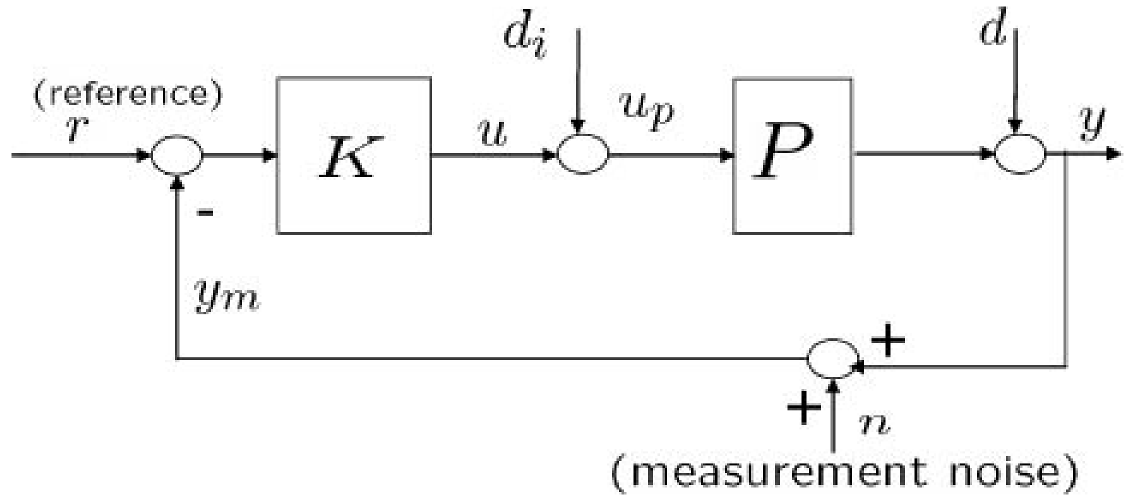


Figure 9:

- $L_i = KP$ is the input loop transfer matrix
- $L_o = PK$ is the output loop transfer matrix
- $S_i = (I + L_i)^{-1}$ is the transfer matrix from d_i to u_p is the input sensitivity matrix.

- $S_o = (I + L_o)^{-1}$ is the output sensitivity matrix.
- $T_i = L_i(I + L_i)^{-1}$ is the input complimentary sensitivity matrix
- $T_o = L_o(I + L_o)^{-1}$ is the output complimentary sensitivity matrix.

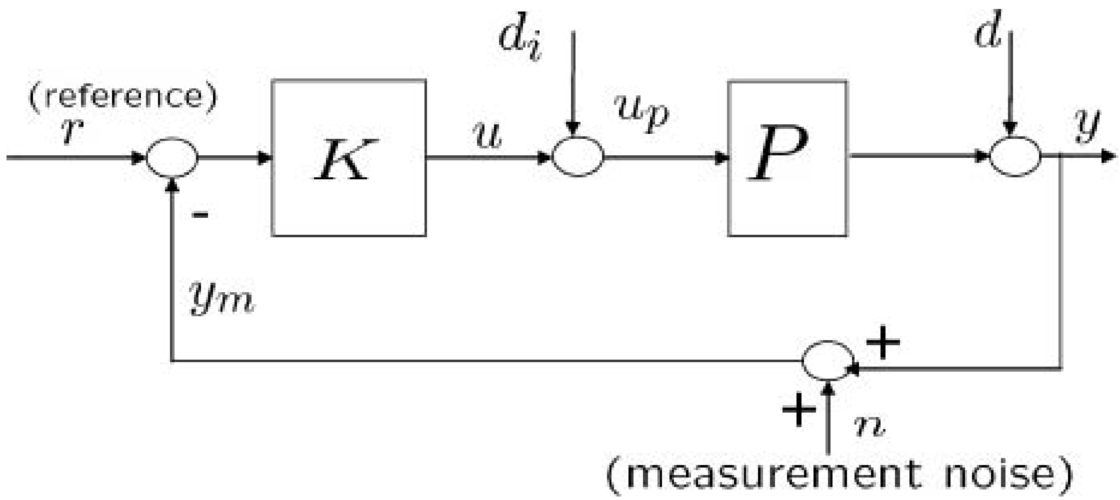


Figure 10:

Loop equations are given by

$$
\begin{aligned}
y &= T_o(r-n) + S_oPd_i + S_od \\
r - y &= S_o(r-d) + T_on - S_oPd_i \\
u &= KS_o(r-n) - KS_od - T_id_i \\
u_p &= KS_o(r-n) - KS_od + S_id_i
\end{aligned}
$$

Disturbance Rejection

Loop equations are given by

$$\begin{aligned} y &= T_o(r-n) + S_oPd_i + S_od \\ r-y &= S_o(r-d) + T_on - S_oPd_i \\ u &= KS_o(r-n) - KS_od - T_id_i \\ u_p &= KS_o(r-n) - KS_od + S_id_i \end{aligned}$$

- Good output disturbance rejection at the output y would require small

$$\bar{\sigma}(S_o) = \bar{\sigma}[(I+PK)^{-1}] = \frac{1}{\underline{\sigma}(I+PK)}$$

- Good input disturbance rejection at the output y would require small

$$\bar{\sigma}(S_oP) = \bar{\sigma}[(I+PK)^{-1}P] = \bar{\sigma}(PS_i)$$

- Good input disturbance rejection at the plant input u_p would require small

$$\bar{\sigma}(S_i) = \bar{\sigma}[(I+KP)^{-1}] = \frac{1}{\underline{\sigma}(I+KP)}$$

- Good output disturbance rejection at the plant input u_p would require small

$$\bar{\sigma}(S_i K) = \bar{\sigma}[(I+KP)^{-1}K] = \bar{\sigma}(KS_o)$$

Note that

$$\underline{\sigma}(A) - 1 \leq \underline{\sigma}(I+A) \leq \underline{\sigma}(A) + 1.$$

It follows that

$$\begin{array}{l} \frac{1}{\underline{\sigma}(PK)+1} \leq \frac{1}{\underline{\sigma}(PK+I)} = \bar{\sigma}(S_o) \leq \frac{1}{\underline{\sigma}(PK)-1} \text{ if } \underline{\sigma}(PK) > 1 \\ \frac{1}{\underline{\sigma}(KP)+1} \leq \frac{1}{\underline{\sigma}(KP+I)} = \bar{\sigma}(S_i) \leq \frac{1}{\underline{\sigma}(KP)-1} \text{ if } \underline{\sigma}(KP) > 1 \end{array}$$

It follows that $\bar{\sigma}(S_0)$ and $\bar{\sigma}(S_i)$ are small if and only if $\underline{\sigma}(PK)$ and $\underline{\sigma}(KP)$ are respectively large.

- Thus for good output disturbance rejection at the output one needs $\underline{\sigma}(PK) >> 1$
- Thus for good input disturbance rejection at the plant input one needs $\underline{\sigma}(KP) >> 1$

Now if its assumed that P and K are invertible and that $\underline{\sigma}(PK) >> 1$ then it follows that

$$\begin{array}{rcl} \bar{\sigma}(S_o P) = \bar{\sigma}((I+PK)^{-1}P)) & = & \bar{\sigma}((I+PK)^{-1}PKK^{-1})) \\ & = & \bar{\sigma}((I+PK)^{-1}PK))\bar{\sigma}(K^{-1}) \\ & \approx & \bar{\sigma}(K^{-1}) \\ & = & \frac{1}{\underline{\sigma}(K)} \end{array}.$$

Now $\bar{\sigma}(S_oP)$ has to be small for rejection of input disturbance at the output. Thus

- good input and output rejection at the output $\iff$ $\underline{\sigma}(PK) >> 1$ and $\underline{\sigma}(K) >> 1$ in the appropriate frequency range.

Similarly if its assumed that P and K are invertible and that $\underline{\sigma}(KP) >> 1$ then it follows that

$$\begin{aligned} \bar{\sigma}(S_iK) = \bar{\sigma}((I+KP)^{-1}K)) &= \bar{\sigma}((I+KP)^{-1}KPP^{-1})) \\ &= \bar{\sigma}((I+KP)^{-1}KP))\bar{\sigma}(P^{-1}) \\ &\approx \bar{\sigma}(P^{-1}) \\ &= \frac{1}{\underline{\sigma}(P)}. \end{aligned}$$

Now $\bar{\sigma}(S_iK)$ has to be small for rejection of output disturbance at the input. Thus

- good input and output rejection at the input $\Longleftrightarrow$ $\underline{\sigma}(KP) >> 1$ and $\underline{\sigma}(P) >> 1$ in the appropriate frequency range.

The condition that $\underline{\sigma}(P) >> 1$ is a fundamental limitation in the sense that no controller can alleviate the situation if its not met.

Noise Rejection

Loop equations are given by

$$
\begin{aligned}
y &= T_o(r-n) + S_oPd_i + S_od \\
r-y &= S_o(r-d) + T_on - S_oPd_i \\
u &= KS_o(r-n) - KS_od - T_id_i \\
u_p &= KS_o(r-n) - KS_od + S_id_i
\end{aligned}
$$

- Good noise rejection at the output requires $\bar{\sigma}(T_o) = \bar{\sigma}(L_o(I+L_o)^{-1})$ to be small. This implies that $\bar{\sigma}(PK) << 1$ in the frequency range where the noise effects are predominant.

Thus a tradeoff has to be struck between good noise rejection and good disturbance rejection. Also note that if $\bar{\sigma}(L_o) << 1$ then $S_o \approx I$ and $KS_o \approx K$. Now the effect of noise on the control output u is given by

$$u = KS_on.$$

Thus

- To prevent the noise from saturating the controller $\bar{\sigma}(K) \leq M$ in the frequency range where the loop gain is small.

Performance Specifications Summarized

In a frequency range $[0, \omega_\ell]$ that characterizes the frequency content of the disturbances and tracking needs

- $\underline{\sigma}(PK) >> 1, \;\; \underline{\sigma}(KP) >> 1, \;\; \underline{\sigma}(K) >> 1.$

In a frequency range $[\omega_u, \infty)$ that characterizes the frequency content of the noise and saturation effects

- $\bar{\sigma}(PK) << 1, \;\; \bar{\sigma}(KP) << 1, \;\; \bar{\sigma}(K) \leq M.$

Linear Fractional Transformations

Lower Linear Fractional Transformations

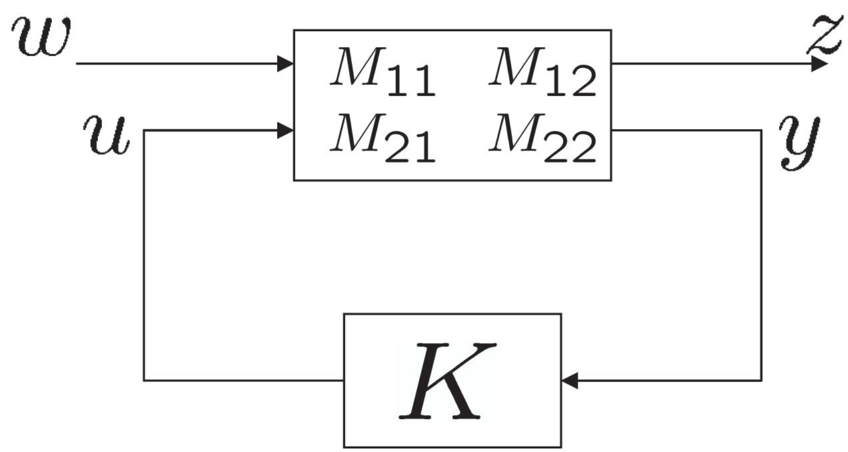


Figure 11:

Suppose

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix}$$

and suppose

$$u = Ky.$$

Then the map from $w \rightarrow z$ is given by

$$\mathcal{F}_\ell(M, K) = M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21}$$

called the lower fractional transformation of M and K.

Upper Linear Fractional Transformations

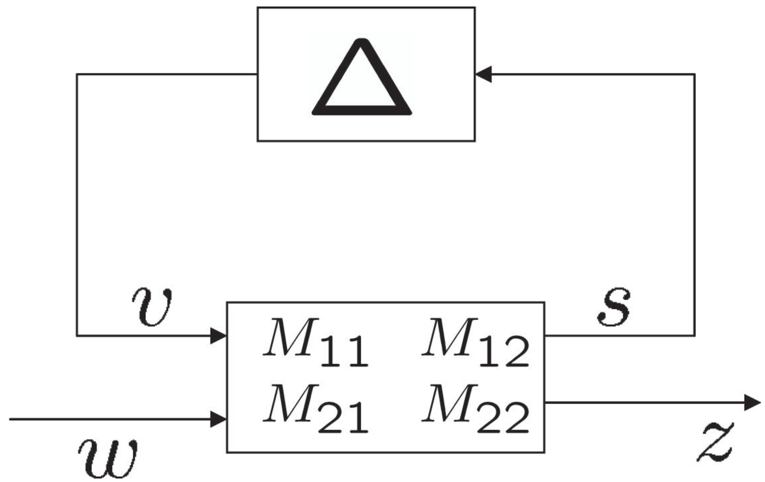


Figure 12:

Suppose

$$\begin{pmatrix} s \\ z \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

and suppose

$$v = \Delta s.$$

Then the map from $w \to z$ is given by

$$\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$$

called the upper fractional transformation of M and Δ.

$G - K - \Delta$ Framework

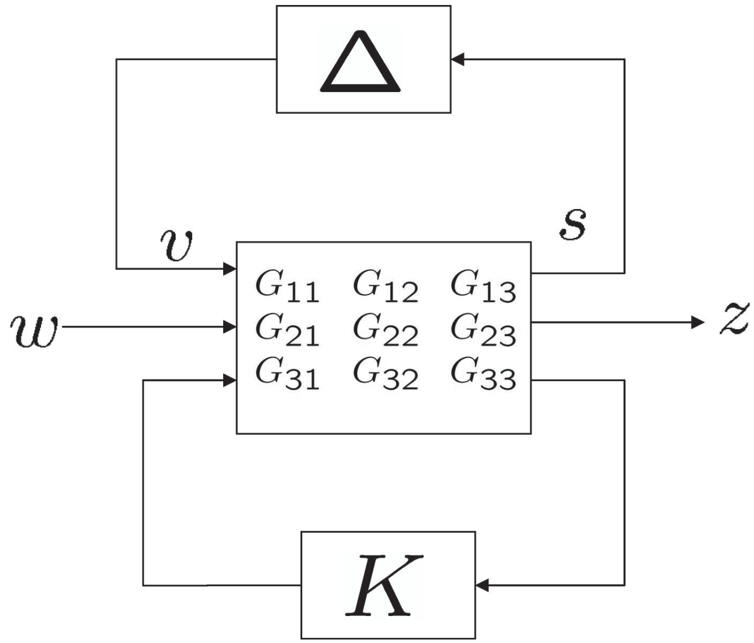


Figure 13:

Note in the above map

$$z = \mathcal{F}_u(\mathcal{F}_\ell(G, K), \Delta)w.$$

The following lemma follows from simple algebra

Lemma 22. *Suppose C is invertible. Then*

$$\begin{aligned}(A+BQ)(C+DQ)^{-1} &= \mathcal{F}_\ell(M,Q)\\ (C+DQ)^{-1}(A+QB)^{-1} &= \mathcal{F}_u(N,Q)\end{aligned}$$

where

$$M=\begin{pmatrix} AC^{-1} & B-AC^{-1}D \\ C^{-1} & -C^{-1}D \end{pmatrix},\quad N=\begin{pmatrix} C^{-1}A & C^{-1} \\ B-DC^{-1}A & -DC^{-1} \end{pmatrix}.$$

Lemma 23. *Let* $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ *and* M_{22} *be invertible. Then*

$$(\mathcal{F}_u(M, \Delta))^{-1} = \mathcal{F}_\ell(N, \Delta)$$

where N *is given by*

$$N = \begin{pmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{pmatrix}.$$

Parametric Uncertainty: An Example

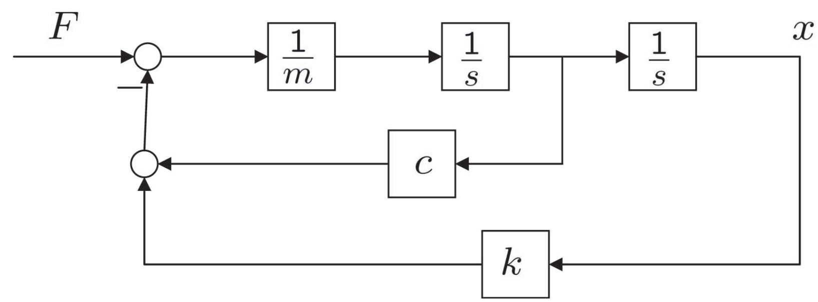


Figure 14:

Consider a spring-mass-damper system where the spring constant is k, the mass m, and the damping factor is c. The dynamical equation is given by

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}$$

as describe by Figure 14.

Suppose k, m and c are each uncertain by 1% of their nominal values $\bar{k}$, $\bar{m}$, and $\bar{c}$. Thus

$$k = \bar{k}(1 + 0.1\delta_k), \quad \frac{1}{m} = \frac{1}{\bar{m}(1 + 0.1\delta_m)} \text{ and } c = \bar{c}(1 + 0.1\delta_c).$$

Note that $\frac{1}{\bar{m}(1+0.1\delta_m)} = \mathcal{F}_\ell(M_1, \delta_m)$ where

$$M = \begin{pmatrix} \frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 1 & -0.1 \end{pmatrix}.$$

The block diagram in terms of the uncertainties $\delta_k,\ \delta_m,$ and δ_c is given in Figure 15.

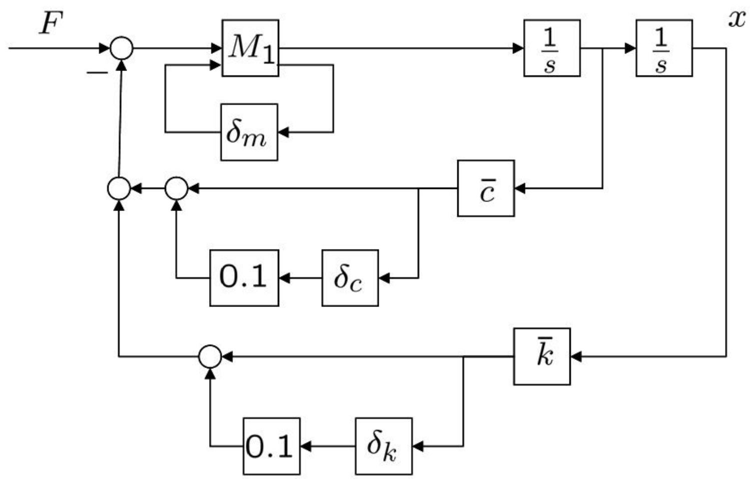


Figure 15:

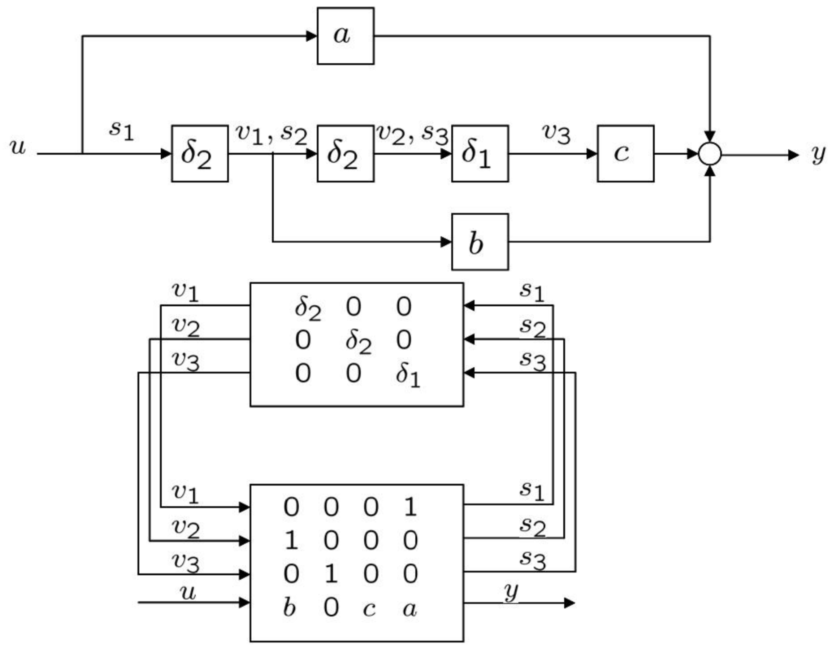
It can be verified that

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \mathcal{F}_\ell(M, \Delta) \begin{pmatrix} x_1 \\ x_2 \\ F \end{pmatrix}$$

where

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 0.1\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{pmatrix} \quad \Delta = \begin{pmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{pmatrix}.$$

Polynomial dependence on uncertain parameters



$$y = (a + b\delta_2 + c\delta_1\delta_2^2)u = \mathcal{F}_\ell(\Gamma_n, \Delta_n)u$$

Figure 16:

Polynomial dependence on uncertain parameters

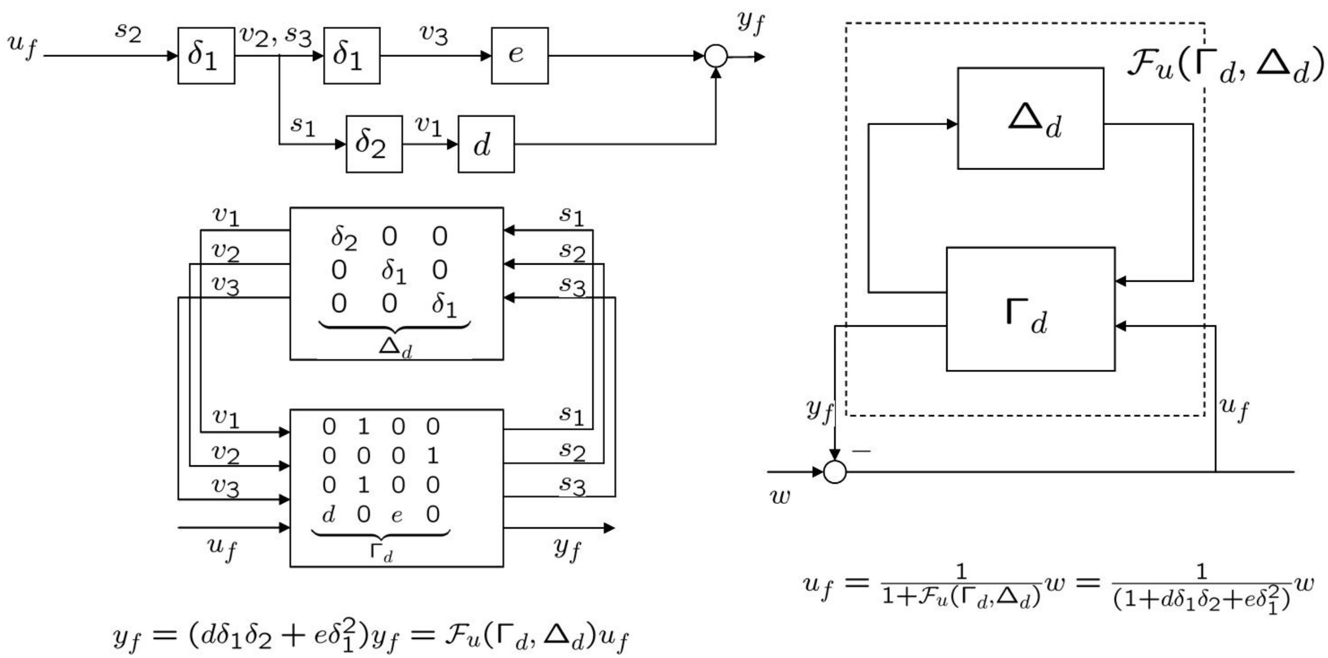
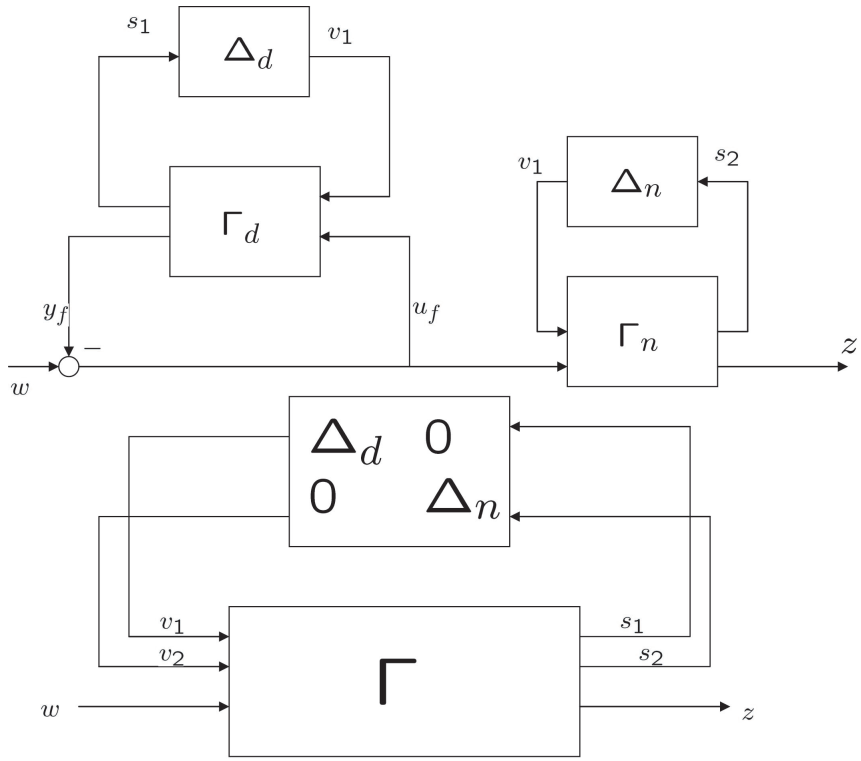


Figure 17:

Rational dependence on uncertain parameters



$$z = \frac{(a+b\delta_2+c\delta_1+c\delta_1\delta_2^2)}{1+d\delta_1\delta_2+e\delta_1^2} = \mathcal{F}_u(\Gamma,\Delta)w$$

Figure 18:

A General Uncertainty Description

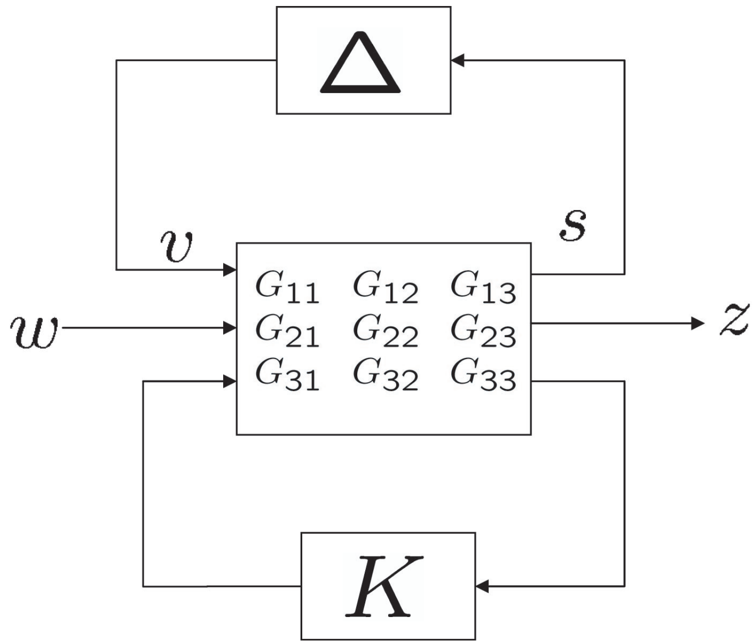


Figure 19:

Note in the above map

$$z = \mathcal{F}_u(\mathcal{F}_\ell(G, K), \Delta)w.$$

We have seen that a general description of the uncertainty is well captured by

$$\Delta \in \{\text{diag}[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}, \Delta_1, \Delta_2, \ldots, \Delta_F] : \delta_i \in \mathcal{R}, \quad \Delta_i \in \mathcal{RH}_\infty\}.$$

The allowable class of uncertainty is

$$\mathbf{\Delta}_{LTI} = \{\text{diag}[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}, \Delta_1, \Delta_2, \ldots, \Delta_F] : \delta_i \in R, \quad \Delta_i \in \mathcal{RH}_\infty\}.$$

Associated with the above class of allowable perturbations we also define

$$\mathbf{B\Delta}_{LTI} = \{\Delta \in \mathbf{\Delta}_{LTI} : \|\Delta\|_\infty \leq 1\}.$$

Note that for any $\Delta \in \mathbf{B\Delta}_{LTI}$ with $\Delta = \text{diag}[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}, \Delta_1, \Delta_2, \ldots, \Delta_F]$ the following conditions are equivalent

- $\|\Delta\|_\infty \leq 1$

- $|\delta_i| \leq 1$ for all $i = 1, \ldots, s$ and $\|\Delta_i(s)\|_\infty \leq 1$ for all $i = 1, \ldots, F$.

We also define the following sets of constant matrices

$$\begin{array}{rcl} \mathbf{\Delta} & = & \{\text{diag}[\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}, \Delta_1, \Delta_2, \ldots, \Delta_F] : \delta_i \in R, \;\; \Delta_i \in \mathcal{C}^{m_j \times m_j}\} \\ \mathbf{B\Delta} & = & \{\Delta \in \mathbf{\Delta} : \bar{\sigma}(\Delta) \leq 1.\} \end{array}$$

Lemma 24. *Given a constant matrix $\Delta \in \mathbf{\Delta}$ and $\omega \in R$ there exists a transfer matrix $\Delta'(s) \in \mathbf{B\Delta}_{LTI}$ such that*

$$\Delta = \Delta'(j\omega).$$

Robust Stability of MIMO Systems

Nominal Stability

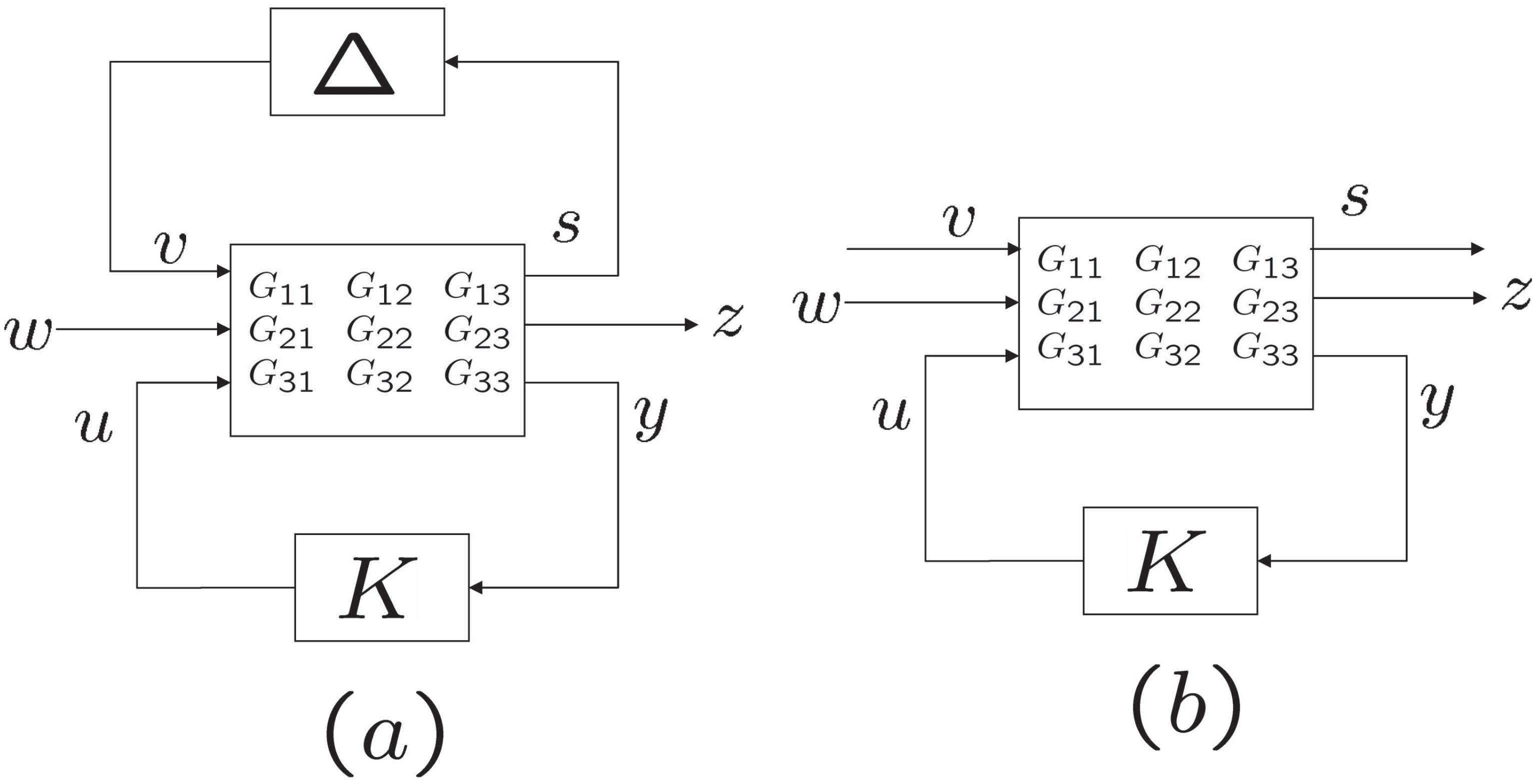


Figure 20:

Definition 51. *The $G - K - \Delta$ interconnection in Figure 20(a) is Nominally stable (NS) if the $G - K$ interconnection in Figure 20(b) is internally stable.*

- Note that the interconnection in Figure 20(b) can be internally stabilized if G can be stabilized through the control input u. In other words if $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization of G then the inherited realization of G_{33} has to be stabilizable and detectable. Otherwise there will be no controller that can internally stabilize the interconnection and thus no controller can achieve nominal stability.

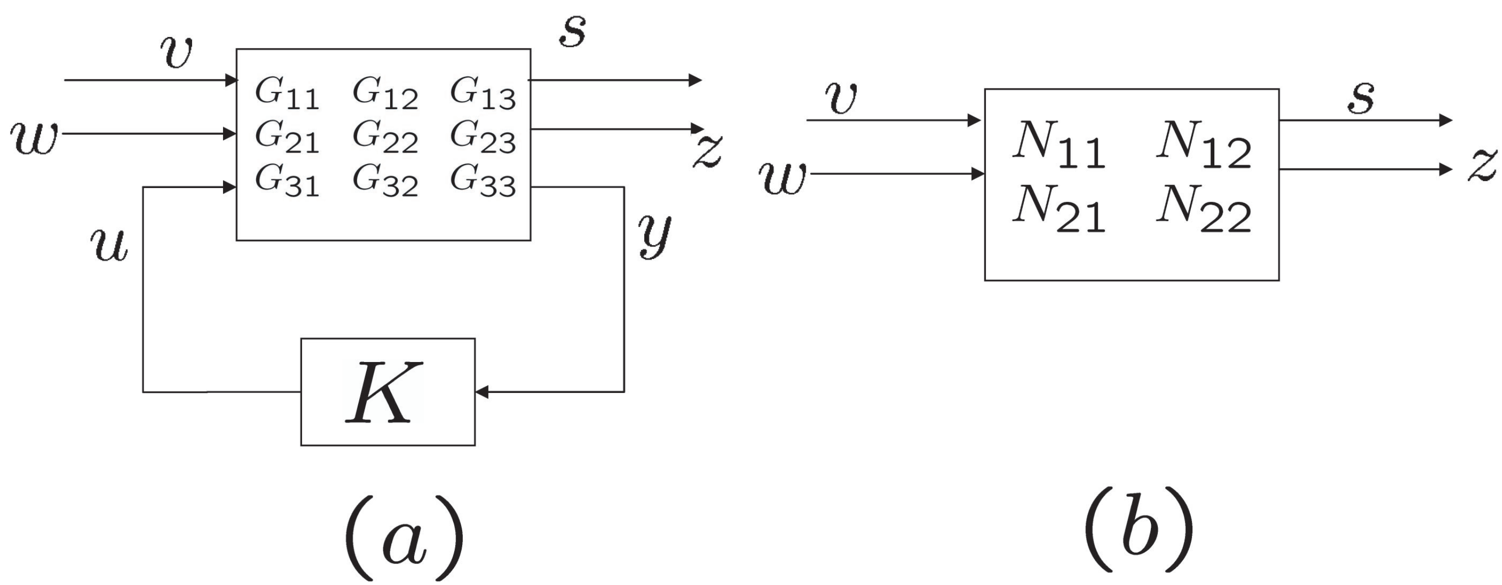


Figure 21:

- Suppose the interconnection in Figure 21(a) is internally stable. That is we have nominal stability. Then

- $N = F_\ell(G, K) = G_{11} + G_{12}K(I - P_{22}K)^{-1}P_{21}$ will be stable.

The $N - \Delta$ Interconnection

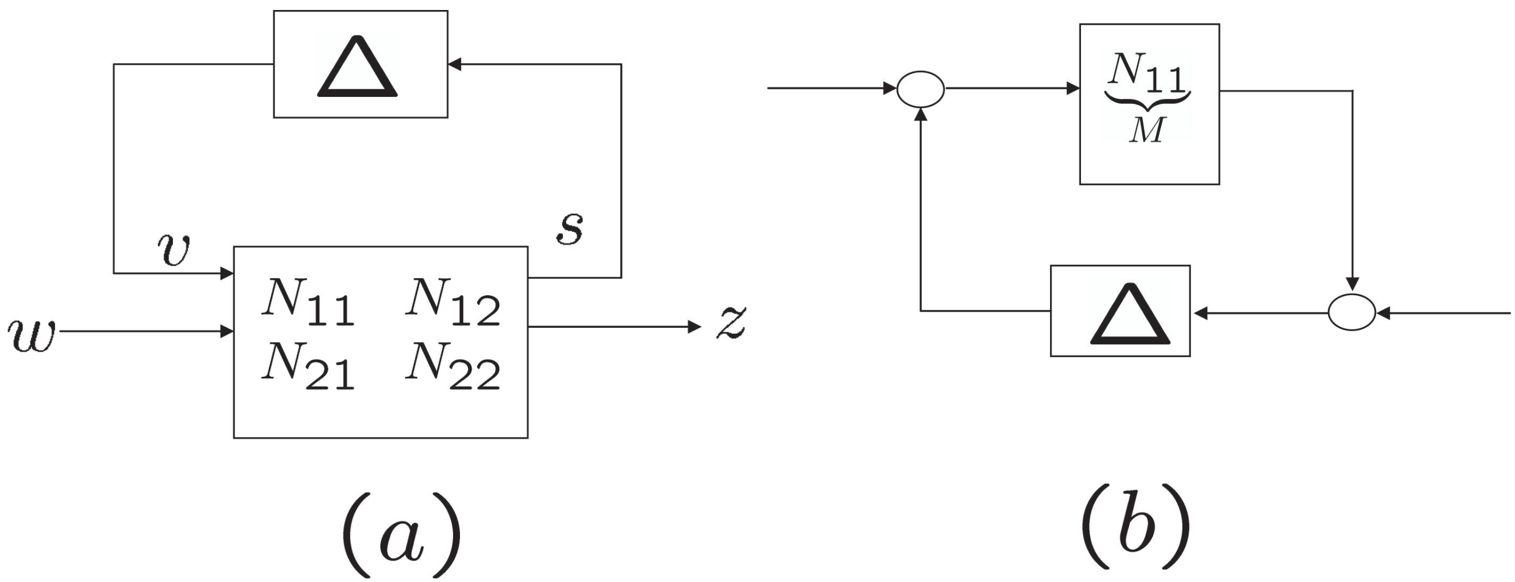


Figure 22:

Suppose the $G - K$ interconnection is internally stable. Then it follows that the $G - K - \Delta$ interconnection is nominally stable and N is stable with $N = \mathcal{F}_\ell(G, K)$.

Now consider the $N - \Delta$ interconnection shown in Figure 22(a). It is evident that the $N - \Delta$ interconnection is stable if and only if $N_{11} - \Delta$ interconnection in Figure 22(b) is stable. This follows as N is stable and therefore N trivially stabilizable through v.

The Robust Stability

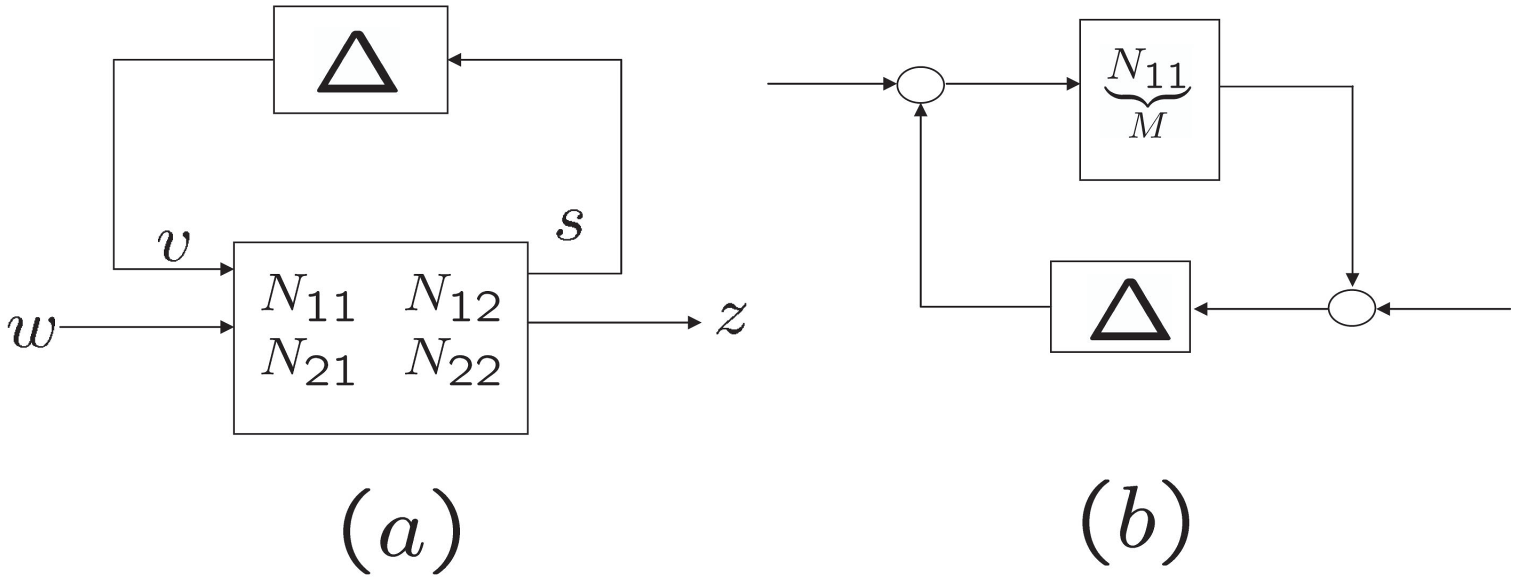


Figure 23:

Definition 52. *The $G - K - \Delta$ interconnection is said to be robustly stable for all $\Delta \in \mathbf{B\Delta}_{LTI}$ if the interconnection is internally stable for all $\Delta \in \mathbf{B\Delta}_{LTI}$.*

From the discussion above it follows that the following two statements are equivalent

- The $G - K - \Delta$ interconnection is robustly stable

- The $G - K$ interconnection is internally stable (Nominal Stability) and the $M - \Delta$ interconnection is internally stable for all $\Delta \in \mathbf{B\Delta}_{LTI}$ with $M = N_{11}$.

A Robust Stability Theorem

Theorem 38. *Assume that M is a stable transfer matrix. The following statements are equivalent.*

- *The $M - \Delta$ interconnection is robustly stable with respect to $\mathbf{\Delta}_{LTI}$ (that is the $M - \Delta$ interconnection is internally stable for all $\Delta \in \mathbf{B\Delta}_{LTI}$).*
- $det(I - M(j\omega)\Delta(j\omega) \neq 0$ *for all* $\Delta \in \mathbf{B\Delta}_{LTI}$ *and for all* $\omega \in R$.

Proof: For $\Delta \in \mathbf{B\Delta}_{LTI}$, Δ is stable. Note that as M and Δ both are stable we have from the Nyquist criterion for MIMO systems the $M - \Delta$ interconnection is stable if and only if the Nyquist contour of $det(I - M(j\omega)\Delta(j\omega))$ does not encircle the origin and does not touch the origin.

$(1 \Rightarrow 2)$ This follows easily from the Nyquist criterion. Note that the Nyquist criterion states that the contour of $det(I - M(j\omega)\Delta(j\omega))$ should not touch the origin for stability. From (1) we have that the $M - \Delta$ interconnection is stable for all Δ in $\mathbf{B\Delta}_{LTI}$ it follows that

$$det(I - M(j\omega)\Delta(j\omega)) \neq 0 \text{ for all } \Delta \in \mathbf{B\Delta}_{LTI} \text{ and for all } \omega \in R.$$

$(2 \Rightarrow 1)$ Suppose there exists a $\Delta \in \mathbf{B\Delta}_{LTI}$ such that the $M - \Delta$ interconnection is not internally stable. From the Nyquist criterion atleast one of the following conditions have to be violated

- $det(I - M(j\omega)\Delta(j\omega)) = 0$ for some $\omega \in R$.
- The Nyquist contour of $det(I - M(j\omega)\Delta(j\omega))$ encircles the origin at least once.

If the first condition holds then the statement is proven. Suppose not. Then the Nyquist contour of $det(I - M(j\omega)\Delta(j\omega))$ encircles the origin at least once.

Note that if $f(\epsilon, s) := det(I - M(s)\epsilon\Delta(s)$ then the Nyquist contour of $f(\epsilon, s)$ changes continuously with respect to ϵ. For $\epsilon = 1$ the Nyquist contour of f encircles 0. For $\epsilon = 0$ the Nyquist contour is a single point 1. Thus for some $\epsilon' \in [0, 1]$ and some $\omega' \in R$, $f(\epsilon', \omega') = 0$ that is $det(I - M(j\omega')\epsilon'\Delta(j\omega')) = 0$. Its evident that $\Delta'(s) := \epsilon'\Delta(s) \in \mathbf{B\Delta}_{LTI}$. Thus there exists a $\Delta' \in \mathbf{B\Delta}_{LTI}$ such that for some ω', $det(I - M(j\omega')\Delta'(j\omega')) = 0$.

■

The following Corollary follows from the theorem above and Lemma 24

Corollary 4. *Assume that M is a stable transfer matrix. The following statements are equivalent.*

- *The $M - \Delta$ interconnection is robustly stable with respect to $\mathbf{\Delta}_{LTI}$.*
- $det(I - M(j\omega)\Delta) \neq 0$ *for all* $\Delta \in \mathbf{B\Delta}$ *and for all* $\omega \in R$.

A Robust Stability for Unstructured Uncertainty

Consider the uncertainty class

$$\{\Delta \in \mathcal{RH}_\infty | \|\Delta\|_\infty \leq 1\}.$$

There is no structure to the class above. It is relatively easy to obtain necessary and sufficient conditions for robust stability with respect to the above class.

Lemma 25. *Let A be a complex matrix. Then it follows that*

$$\max_{\bar{\sigma}(B)\leq 1} \rho(AB) \leq \max_{\bar{\sigma}(B)\leq 1} \bar{\sigma}(AB) = \bar{\sigma}(A).$$

Proof: Note that as $\bar{\sigma}(\cdot)$ is an induced norm it follows that

$$\rho(AB) \leq \bar{\sigma}(AB)$$

and thus

$$\max_{\bar{\sigma}(B)\leq 1} \rho(AB) \leq \max_{\bar{\sigma}(B)\leq 1} \bar{\sigma}(AB).$$

Now suppose the singular value decomposition of A is given by $U\Sigma V^*$. Let $B' := VU^*$. Then it follows that

$$\bar{\sigma}(B') \leq \bar{\sigma}(V)\bar{\sigma}(U^*) = 1.$$

Furthermore we have that

$$\bar{\sigma}(AB') = \bar{\sigma}(U\Sigma V^* V U^*) = \bar{\sigma}(U\Sigma V^*) = \bar{\sigma}(\Sigma) =: \sigma_1.$$

Now

$$AB' = U\Sigma U^*$$

Thus it follows that

$$\rho(AB') = \sigma_1 = \bar{\sigma}(A).$$

Thus we have constructed a matrix B' with $\sigma bar(B') \leq 1$ and $\rho(AB') = \bar{\sigma}(AB')$. Thus it follows that

$$\max_{\bar{\sigma}(B)\leq 1} \rho(AB) \geq \bar{\sigma}(A).$$

This proves the lemma. ■

Theorem 39. *Let $M(s)$ be a stable transfer matrix. The $M - \Delta$ interconnection is internally stable for all $\Delta \in \{\Delta' \in \mathcal{RH}_\infty : \|\Delta\|_\infty \leq 1\}$ if and only if $\|M(s)\|_\infty < 1$.*

Proof: $(\Leftarrow)$ Suppose $\|M\|_\infty < 1$. Let $\Delta \in \{\Delta' \in \mathcal{RH}_\infty : \|\Delta\|_\infty \leq 1\}$. Let $\omega \in R$. Then

$$\rho(M(j\omega)\Delta(j\omega)) \leq \bar{\sigma}(M(j\omega)\Delta(j\omega)) \leq \bar{\sigma}(M(j\omega))\bar{\sigma}(\Delta(j\omega)) < 1.$$

Thus it follows that

$$det(I - M(j\omega)\Delta(j\omega)) \neq 0 \text{ for all } \omega \in R.$$

Thus from Theorem 38 that the $M - \Delta$ interconnection is robustly stable with respect to the class

$$\{\Delta' \in \mathcal{RH}_\infty : \|\Delta\|_\infty \leq 1\}.$$

$(\Rightarrow)$ Suppose $\|M\|_\infty \geq 1$ and suppose ω is such that $\bar{\sigma}(M(j\omega)) \geq 1$. It follows from Lemma 25

$$\max_{\bar{\sigma}(\Delta)\leq 1} \rho(M(j\omega)\Delta) = \bar{\sigma}(M(j\omega)) \geq 1.$$

Thus there exists a constant matrix Δ with $\bar{\sigma}(\Delta) \leq 1$ such that

$$M(j\omega)\Delta x = \lambda x$$

with $x \neq 0$ and $|\lambda| \geq 1$. Let

$$\Delta' = \frac{1}{\lambda}\Delta.$$

Note that $\bar{\sigma}(\Delta') \leq 1$ Then it follows that $M(j\omega)\Delta'$ has an eigenvalue at 1 and thus $det(I - M(j\omega)\Delta') = 0$. One can construct a $\Delta(s) \in \mathcal{RH}_\infty$ with the property that $\Delta(j\omega) = \Delta'$ and $\|\Delta\|_\infty \leq 1$. Thus we have constructed a $\Delta(s)$ with $\|\Delta\|_\infty \leq 1$ with

$$det(I - M(j\omega)\Delta(j\omega)) = det(I - M(j\omega)\Delta') = 0.$$

From Theorem 38 it follows that the $M - \Delta$ interconnection is not robustly stable.

■

A General Uncertainty Description

- Let the structure of the uncertainty structure be captured by the class $\mathbf{\Delta}$.

- Define $\mu_{\mathbf{\Delta}} : \mathcal{C}^{n\times n} :\rightarrow R$ by

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\bar{\sigma}(\Delta) : det(I - M\Delta) = 0 \text{ with } \Delta \in \mathbf{\Delta}\}}.$$

The following theorem elucidates the significance of this definition.

Theorem 40. *Suppose the class of allowable uncertainty is given by* $\mathbf{\Delta}$. *Then the* $M - \Delta$ *interconnection is stable for all* $\Delta(s) \in \mathbf{B\Delta}_{LTI}$ *if and only if*

$$\mu_{\mathbf{\Delta}}(M(j\omega)) < 1 \text{ for all } \omega.$$

Proof: From Corollary 4 that the $M - \Delta$ interconnection is stable if and only if for any $\omega \in R$

$$\begin{aligned}
& det(I - M(j\omega)\Delta) \neq 0 \text{ for all } \Delta \in \mathbf{B\Delta} \\
\Leftrightarrow \quad & [det(I - M(j\omega)\Delta) = 0 \text{ for any } \Delta \in \mathbf{\Delta}] \Rightarrow \bar{\sigma}(\Delta) > 1 \\
\Leftrightarrow \quad & \min_{\Delta \in \mathbf{\Delta}}\{\bar{\sigma}(\Delta) : det(I - M(j\omega)\Delta) = 0 \} > 1 \qquad . \\
\Leftrightarrow \quad & \frac{1}{\min_{\Delta \in \mathbf{\Delta}}\{\bar{\sigma}(\Delta) : det(I - M(j\omega)\Delta) = 0 \}} < 1 \\
\Leftrightarrow \quad & \mu_{\mathbf{\Delta}}(M(j\omega)) < 1.
\end{aligned}$$

This proves the theorem. ■

Properties of μ.

1. For any uncertainty structure $\mathbf{\Delta}$ and scalar α, $\mu(\alpha M) = |\alpha|\mu(M)$.

2. For any uncertainty structure $\mathbf{\Delta}$

$$\mu(M) \le \bar{\sigma}(M).$$

3. If the uncertainty structure $\mathbf{\Delta}$ is such that it consists only of full complex blocks $\mathbf{\Delta} = \{\Delta \in \mathcal{C}^{n\times n}\}$ then

$$\mu(M) = \bar{\sigma}(M).$$

4. Let $\mathbf{D}$ be a set of matrices that commute with the matrices in $\mathbf{\Delta}$ (that is if $D \in \mathbf{\Delta}$ then $D\Delta = \Delta D$ for all $\Delta \in \mathbf{\Delta}$) then for any $D \in \mathbf{D}$

$$\mu(M) = \mu(DMD^{-1}).$$

5. Let $\mathbf{D}$ be a set of matrices that commute with the matrices in $\mathbf{\Delta}$ (that is if $D \in \mathbf{\Delta}$ then $D\Delta = \Delta D$ for all $\Delta \in \mathbf{\Delta}$) then for any $D \in \mathbf{D}$

$$\mu(M) \leq \bar{\sigma}(DMD^{-1}).$$

6. If the uncertainty structure consists only of complex blocks then

$$\mu(M) = \max_{\Delta \in \mathbf{B\Delta}} \rho(M\Delta).$$

7. For any uncertainty structure $\mathbf{\Delta}$ and for any unitary matrix $U \in \mathbf{\Delta}$

$$\mu(MU) = \mu(M) = \mu(UM).$$

8. Let $\mathbf{\Delta}$ consist only of complex blocks then

$$\mu(M) = \max_{U \in \mathbf{U}} \rho(MU)$$

where

$$\mathbf{U} = \{U : U^*U = I \text{ and } U \in \mathbf{\Delta}\}.$$

Note that from Theorem 40 we have that the $M - \Delta$ interconnection is robustly stable with respect to $\mathbf{\Delta}$ if and only if

$$\mu_\Delta(M(j\omega)) \leq 1 \text{ for all } \omega \in R.$$

This condition can be replaced by

$$\sup_{\omega \in R}[\mu_{\mathbf{\Delta}}(M(j\omega))] < 1.$$

Note $M = N_{11}$ where $N = \mathcal{F}_\ell(G, K)$. Thus robust stability is guaranteed if one can find a controller K that internally stabilizes the $G - K$ interconnection and

$$\sup_{\omega \in R}[\mu_{\mathbf{\Delta}}(M(j\omega))] < 1.$$

Thus the synthesizing the optimal controller will be obtained by solving the following problem:

$$\inf_{K \text{ stabilizing}} \sup_{\omega \in R} [\mu_{\mathbf{\Delta}}(M(K)(j\omega))].$$

Note that computing $\mu_{\mathbf{\Delta}}(M(K)(j\omega))$ is not easy and thus we replace it with its upper bound $\bar{\sigma}(M(K)(j\omega))$. This bound can be improved by using the fact that

$$\mu_{\mathbf{\Delta}}(M(K)(j\omega)) = \mu_{\mathbf{\Delta}}(D(\omega)M(K)(j\omega)D^{-1}(\omega))] \leq \bar{\sigma}[D(\omega)M(K)(j\omega)D^{-1}(\omega)]$$

for all $D(\omega) \in \mathbf{D}$ where $\mathbf{D}$ is the set of complex matrices that commute with matrices in $\mathbf{\Delta}$. Thus we can use the following bound

$$\mu_{\mathbf{\Delta}} M(K)(j\omega) \leq \inf_{D(\omega) \in \mathbf{D}} \bar{\sigma}(D(\omega)M(K)(j\omega)D^{-1}).$$

Thus we have the following problem to solve

$$\inf_{K \text{ stabilizing}} \sup_{\omega} \inf_{D(\omega)\in\mathbf{D}} \bar{\sigma}(D(\omega)M(K)(j\omega)D^{-1}(\omega)).$$

Let $\tilde{\mathbf{D}}$ be the set of maps from $R \to \mathcal{C}^{n\times n}$ such that every element $\tilde{D} \in \tilde{\mathbf{D}}$ satisfies $\tilde{D}(\omega) \in \mathbf{D}$. Thus the problem is

$$\inf_{K \text{ stabilizing}} \sup_{\omega} \inf_{\tilde{D}\in\tilde{\mathbf{D}}} \bar{\sigma}(D(\omega)M(K)(j\omega)D^{-1}(\omega)).$$

It is true that

$$\sup_{\omega} \inf_{\tilde{D}\in\tilde{\mathbf{D}}} \bar{\sigma}(D(\omega)M(K)(j\omega)D^{-1}(\omega)) = \inf_{\tilde{D}\in\tilde{\mathbf{D}}} \sup_{\omega} \bar{\sigma}(D(\omega)M(K)(j\omega)D^{-1}(\omega)).$$

Thus the problem becomes

$$\inf_{K \text{ stabilizing}} \inf_{\tilde{D} \in \tilde{\mathbf{D}}} \sup_{\omega} \bar{\sigma}(D(\omega) M(K)(j\omega) D^{-1}(\omega)).$$

We will use another upper bound. Let

$$\mathbf{D}_s := \{D(s) \in \mathcal{RH}_\infty^{n\times n} : D^{-1}(s) \in \mathcal{RH}_\infty^{n\times n} \text{ and } D(j\omega) \text{ commutes with } \mathbf{\Delta}\}.$$

Note that

$$\begin{aligned} &\inf_{K \text{ stabilizing}} \inf_{\tilde{D} \in \tilde{\mathbf{D}}} \sup_{\omega} \bar{\sigma}(D(\omega) M(K)(j\omega) D^{-1}(\omega)) \\ \leq &\inf_{K \text{ stabilizing}} \inf_{D_s \in \mathbf{D}_s} \sup_{\omega} \bar{\sigma}(D_s(\omega) M(K)(j\omega) D_s^{-1}(\omega)). \end{aligned}$$

We will use the greater upper bound as it is easier to solve.

$$\inf_{K \text{ stabilizing}} \inf_{D_s \in \mathbf{D}_s} \sup_{\omega} \bar{\sigma}(D(j\omega)M(K)(j\omega)D^{-1}(j\omega))$$
$$= \inf_{K \text{ stabilizing}} \inf_{D_s \in \mathbf{D}_s} \|D_s M(K) D_s^{-1}\|_\infty$$

Things to note

- Suppose K is a stabilizing controller. Then

$$\inf_{D_s \in \mathbf{D}_s} \|D_s M(K) D_s^{-1}\|_\infty \tag{28}$$

  can be solved.

- Suppose $D_s \in \mathcal{RH}_\infty^{n \times n}$ is a fixed transfer matrix. Then

$$\inf_{K \text{ stabilizing}} \|D_s M(K) D_s^{-1}\|_\infty \tag{29}$$

is a standard $\mathcal{H}_\infty$ problem and can be solved.

- The joint problem

$$\inf_{K \text{ stabilizing},\ D_s \in \mathbf{D}_s} \|D_s M(K) D_s^{-1}\|_\infty$$

is hard to solve.

The $D - K$ iteration scheme operates by first assuming $D_s = I$. Compute K_1 that solves (28). Then with $K = K_1$ solve the problem (29). Let $D_1(s)$ be the solution. Solve (28) with $D_s = D_1$ to obtain K_2. Iterate to get a satisfactory result. Note that there is no guarantee of convergence for this problem.

Multiple Input Multiple Output Interconnections: Robust Performance

Robust Performance

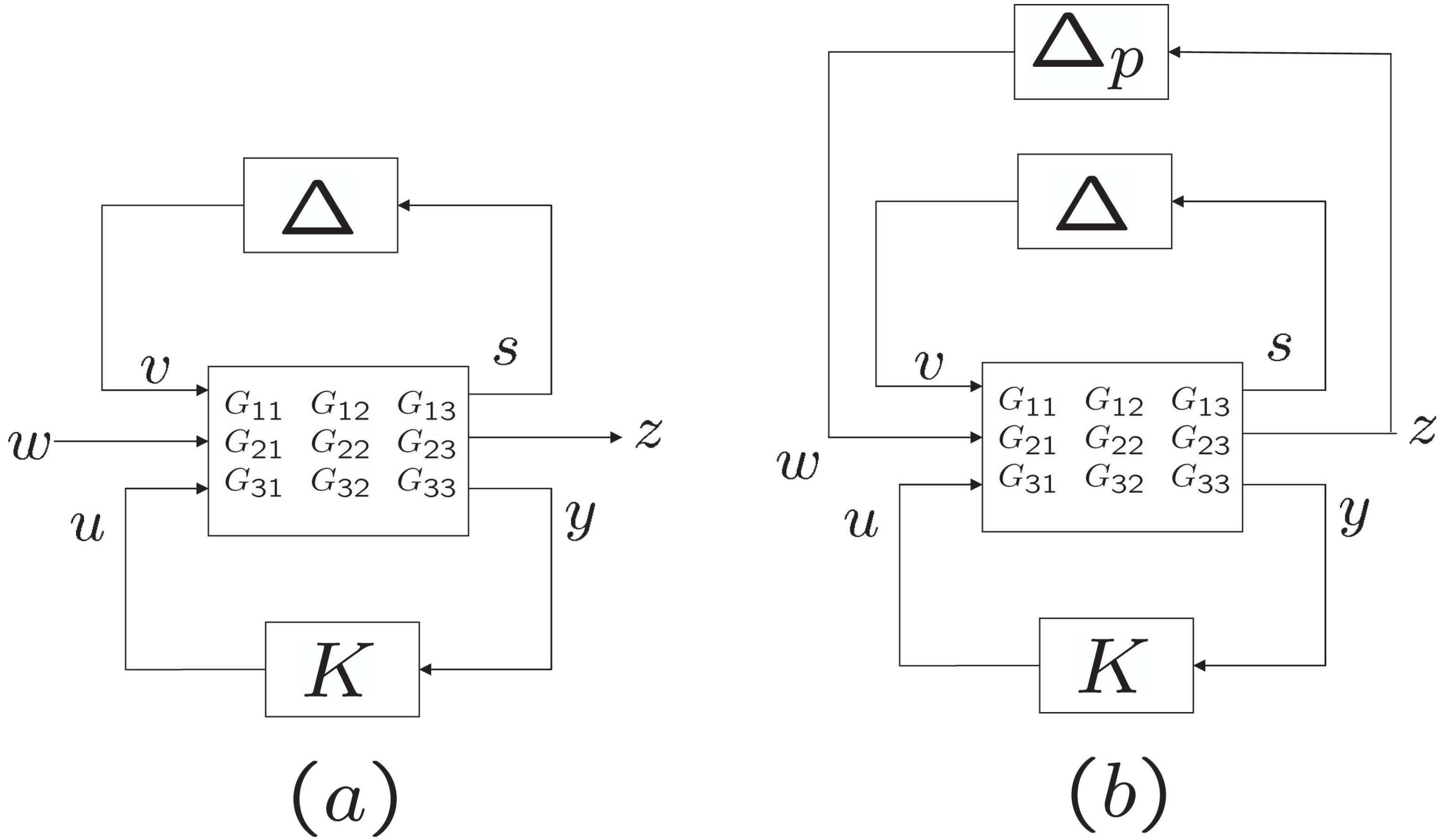


Figure 24:

Definition 53. *The $G-K-\Delta$ interconnection achieves robust performance if the $G-K-\Delta$ is robustly stable with respect to $\mathbf{\Delta}_{LTI}$ and*

$$\|\mathcal{F}_u(\mathcal{F}_\ell(G,K),\Delta)\|_\infty < 1 \text{ for all } \Delta \in \mathbf{B\Delta}_{LTI}.$$

Theorem 41. *The $G-K-\Delta$ interconnection achieves robust performance if and only if the $G-K-\Delta$ interconnection is nominally stable and*

$$\sup_\omega \mu_{\mathbf{\Delta}_P}[\mathcal{F}_\ell(G,K)(j\omega)] < 1$$

where

$$\mathbf{\Delta}_P := \{diag(\Delta_p, \Delta) : \Delta \in \mathbf{\Delta}_{LTI}\},\ \Delta_p \in \mathcal{RH}_\infty^{n_w \times n_z}.$$

Proof: $(\Rightarrow)$ Suppose the $G-K-\Delta$ framework achieves robust performance. Then it follows that $G-K$ interconnection is internally stable (Nominal Stability) and the $N-\Delta$ (with $N=\mathcal{F}_\ell(G,K)$) interconnection is stable for all $\Delta \in \mathbf{B\Delta}_{LTI}$. Let $\Delta \in \mathbf{B\Delta}_{LTI}$ and let $M := \mathcal{F}_u(N,\Delta)$. The $N-\Delta$

interconnection is stable and $\|\mathcal{F}_u(N, \Delta)\|_\infty < 1$. Thus from Theorem 39 (small gain theorem) it follows that the the $M - \Delta_p$ interconnection is stable for any

$$\Delta_p \in \{\Delta_s \in \mathcal{RH}_\infty^{n_w \times n_z} : \|\Delta_s\|_\infty \leq 1\}.$$

This proves that the $M - \Delta_p$ interconnection is internally stable. This implies that the interconnection of $\mathcal{F}_\ell(G, K)$ and any $\Delta_p \in \mathbf{\Delta}_P$ with $\|\Delta_p\|_\infty \leq 1$ is stable. That is $\mathcal{F}_\ell(G, K)$ achieves robust stability with respect to $\mathbf{\Delta}_P$. Thus from Theorem 40

$$\sup_\omega \mu_{\mathbf{\Delta}_P}[\mathcal{F}_\ell(G, K)(j\omega)] < 1.$$

$(\Leftarrow)$ Suppose

$$\sup_\omega \mu_{\mathbf{\Delta}_P}[\mathcal{F}_\ell(G, K)(j\omega)] < 1$$

and $G - K$ interconnection is internally stable (Nominal stability). Then $\mathcal{F}_\ell(G, K)$ is stable and $\mathcal{F}_\ell(G, K) - \Delta_P$ interconnection is internally stable for any $\Delta_P \in \mathbf{B}\mathbf{\Delta}_P$.

Let $\Delta \in \mathbf{\Delta}_{LTI}$. Then the $M - \Delta_p$ interconnection is stable with $M := \mathcal{F}_u(N, \Delta)$ and

$$\Delta_p \in \{\Delta_s \in \mathcal{RH}_\infty^{n_w \times n_z} : \|\Delta_s\|_\infty \leq 1\}.$$

From the small gain theorem on unstructured uncertainty it follows that

$$\|M\|_\infty < 1.$$

Thus

$$\|\mathcal{F}_u(N, \Delta)\|_\infty < 1 \text{ for all } \Delta \in \mathbf{B}\mathbf{\Delta}_{LTI}.$$

■

$\mathcal{H}_\infty$ Loop Shaping

McFarlane Glover Design

McFarlane Glover Design

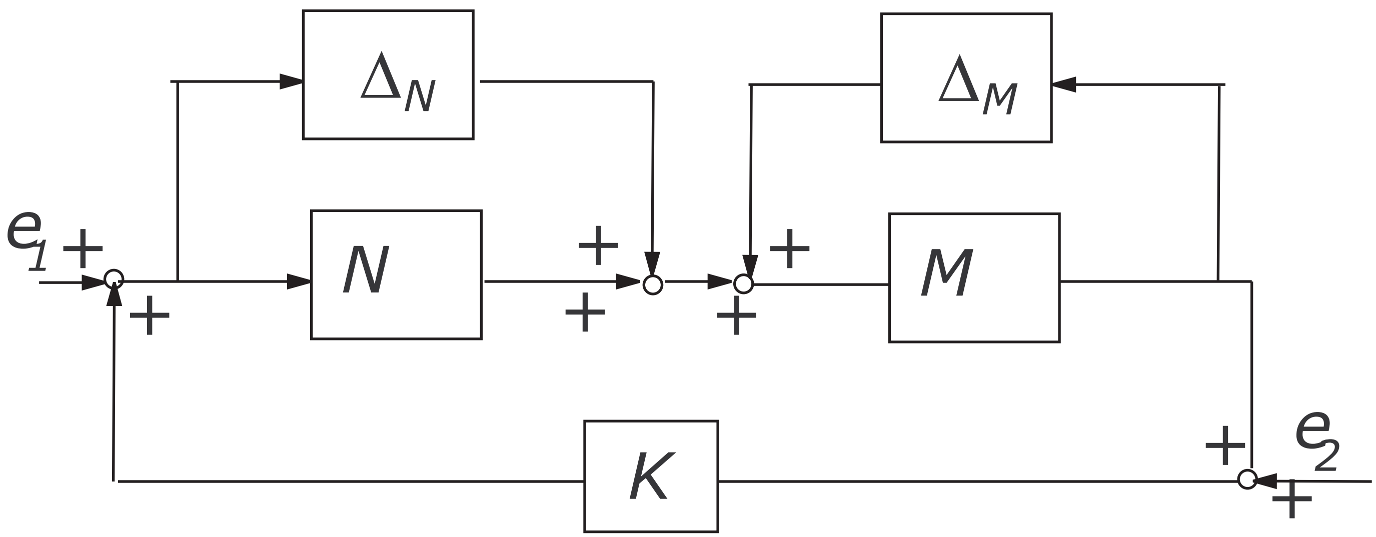


Figure 25:

- The nominal plant is given in the coprime factor form as $G = \tilde{M}^{-1}\tilde{N}$.
- The perturbed plant is given by

$$G_\Delta = G = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)$$

where Δ_M and Δ_N are stable unknown transfer functions.

- The robust design objective is to stabilize not only the nominal model but the family of the perturbed plants given by

$$\mathcal{G}_\epsilon = \{(\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N) : \|[\Delta_M, \Delta_N]\|_{\mathcal{H}_\infty} < \epsilon\}$$

with a controller K as shown in Figure 25.

Advantages of Coprime Factor Perturbation Form

- In the other forms of perturbations (e.g. additive uncertainty, multiplicative uncertainty forms) the number of unstable poles of the nominal and the perturbed plants has to be the same. In the coprime perturbation form the number of unstable poles and zeros for the perturbed form can be different than the number of unstable poles of the nominal plant.

- The solution in this case is particularly elegant.

- It can be used for robustyfying any existing closed loop design.

Preliminaries

Definition 54. *The feedback system of Figure 25 denoted by* $(\tilde{M}, \tilde{N}, K, \epsilon)$ *is robustly stable if and only if the interconnection* (G_Δ, K) *is internally stable for all* $G_\Delta \in \mathcal{G}_\epsilon$.

$M - \Delta$ Structure

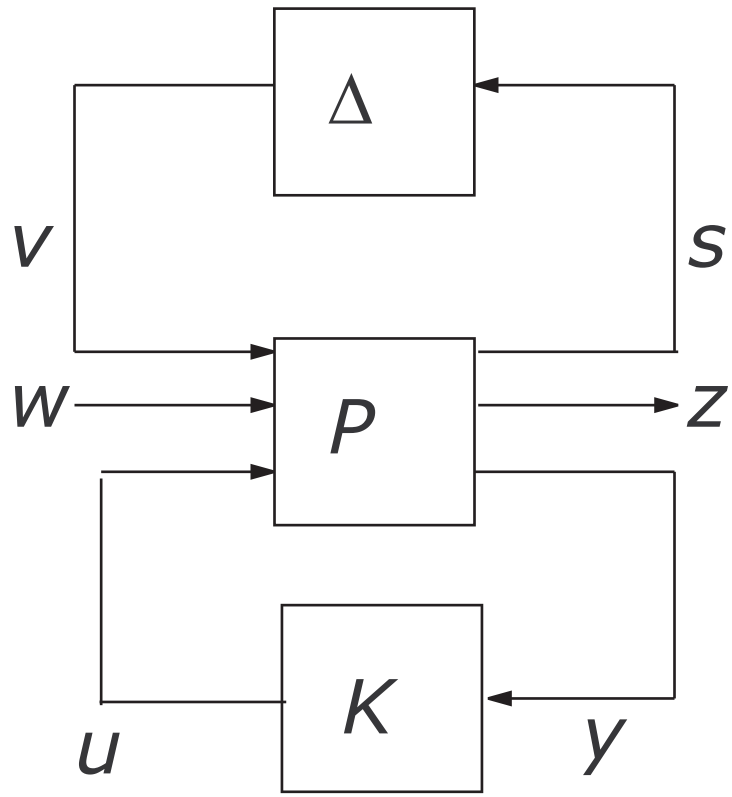


Figure 26:

We will first cast the coprime perturbation robust stability problem into the standard framework shown in Figure 26 with $\Delta := [\Delta_M, \Delta_N]$. Note that in this case the signals w and z are absent and v in Figure 26 corresponds to v in Figure 25. Also, s in Figure 26 is given by the vector $(s_M, \;\; s_N)^T$.

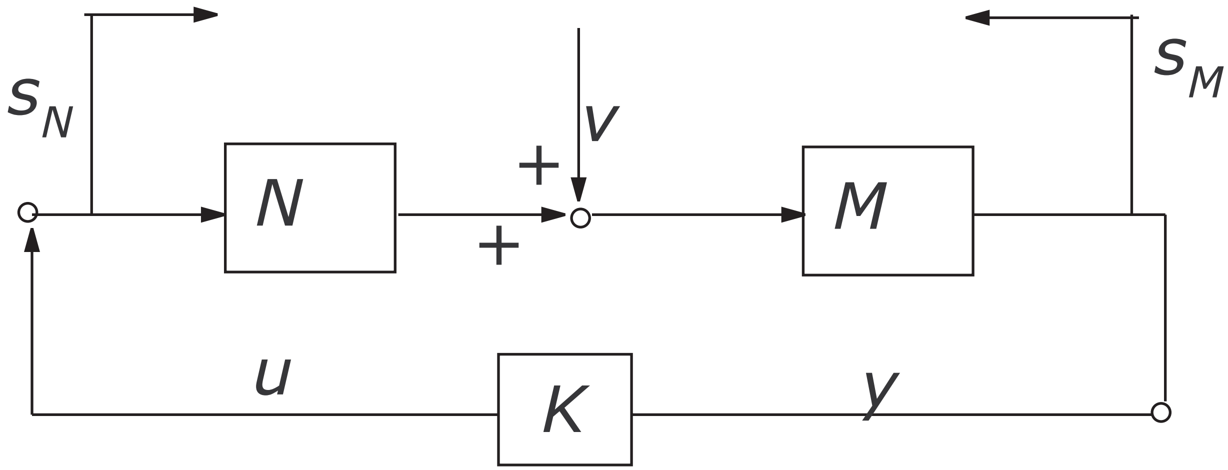


Figure 27:

Note that

$$\begin{pmatrix} s_M \\ s_N \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} M^{-1} & M^{-1}N \\ 0 & I \\ M^{-1} & M^{-1}N \end{pmatrix}}_{P} \begin{pmatrix} v \\ u \end{pmatrix}.$$

$$u = Ky$$

Note that $y = M^{-1}v + M^{-1}Nu = M^{-1}v + Gu = M^{-1}v + GKy$. Thus

$y = (I - GK)^{-1}M^{-1}v$. This implies that $u = K(I - GK)^{-1}M^{-1}v$. Thus we obtain

$$\begin{pmatrix} s_M \\ s_N \end{pmatrix} = \begin{pmatrix} y \\ u \end{pmatrix} = \underbrace{\begin{pmatrix} (I - GK)^{-1}M^{-1} \\ K(I - GK)^{-1}M^{-1} \end{pmatrix}}_{\mathbf{M}} v.$$

Small Gain Theorem

- From the small gain theorem it follows that the interconnection is stable for all $\Delta \in \mathcal{G}_\epsilon$ if and only if

$$\|\mathbf{M}\|_{\mathcal{H}_\infty} = \left\| \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1} M^{-1} \right\|_{\mathcal{H}_\infty} \leq \frac{1}{\epsilon}.$$

- Thus the robust stability problem in the coprime perturbation setting can be solved by an equivalent $\mathcal{H}_\infty$ problem.

- The solution is even more elegant; there is no need for iterations to obtain the optimal controller which achieves the largest ϵ.

Glover McFarland Design

Theorem 42. *A controller K is stabilizing and satisfies*

$$\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1} M^{-1} \right\|_{\mathcal{H}_\infty} \leq \gamma$$

if and only if K has a rcf $K = UV^{-1}$ for some $U, V \in R\mathcal{H}_\infty$ satisfying

$$\left\| \begin{pmatrix} -N^* \\ M^* \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} \right\|_{\mathcal{H}_\infty} \leq (1 - \gamma^{-2})^{\frac{1}{2}}.$$

Theorem 43. *1. Optimal solutions to the normalized lcf robust stabilization problem gives*

$$\inf_{K \text{ stabilizing}} \left\| \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1} M^{-1} \right\|_{\mathcal{H}_\infty} = \{1 - \|[N, M\|_H^2\}^{-\frac{1}{2}}.$$

2. The maximum robust stability margin is

$$\epsilon_{\max} = \{1 - \|[N, M]\|_H^2\}^{\frac{1}{2}}$$

3. All optimal controllers are given by $K = UV^{-1}$ *where* $U, V \in R\mathcal{H}_\infty$ *satisfy*

$$\left\| \begin{pmatrix} -N^* \\ M^* \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} \right\|_{\mathcal{H}_\infty} = \|[N, M\|_H.$$

McFarland Glover Controller

Theorem 44. *The controller K (a positive feedback controller) that guarantees*

$$\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1} M^{-1} \right\|_{\mathcal{H}_\infty} \leq \gamma$$

for a specified $\gamma > \gamma_{\min}$ is given by

$$\left[\begin{array}{c|c} A + BF + \gamma^2 (L^T)^{-1} Z C^T (C + DF) & \gamma^2 (L^T)^{-1} Z C^T \\ \hline B^T X & -D^T \end{array} \right]$$

where

$$\gamma_{\min} = \frac{1}{\epsilon_{\max}} = (1 + \rho(XZ))^{\frac{1}{2}}, \; F = -S^{-1}(D^T C + B^T X), \quad L = (1 - \gamma^2) I + XZ,$$

and X and Z are the solutions to

$$(A - BS^{-1}D^TC)^TX + X(A - BS^{-1}D^TC) - XBS^{-1}B^TX + C^TR^{-1}C = 0 \text{ and} \quad (30)$$

$$(A - BS^{-1}D^TC)Z + Z(A - BS^{-1}D^TC)^T - ZC^TR^{-1}CZ + BS^{-1}B^T = 0 \quad (31)$$

with $R = I + DD^T$ and $S = I + D^TD$.

SISO $\mathcal{H}_2 Problem$

Consider a discrete time generalized system $G(z)$ and suppose that we denote by $\lambda := z^{-1}$. Then $G(z) = G(\lambda)$ is stable if and only if all poles λ are outside the unit disc. The unstable poles and zeros are the ones that are inside the closed unit disc. Suppose the set of closed loop maps achieveable via stabilizing controllers is given by

$$\{H(\lambda) - U(\lambda)Q(\lambda) : \ Q \text{ stable}\}$$

where h, u are stable transfer functions.

Suppose we denote the closed-loop map by $\Phi(q) := H - UQ$. We denote the impulse response of the $H,\ U,\ Q$ and Φ by $h,\ u,\ q$ and ϕ respectively. As all the transfer functions are stable we have that $h,\ u,\ q$ and $\phi \in \ell_1$.

Suppose the input to this system is white with variance σ^2. Then the output variance is given by

$$\sigma^2[\sum_{k=0}^{\infty} \phi(k)^2] =: \sigma^2 \|\phi\|_2^2.$$

Thus if ϕ denotes a system transfer function between a white noise input with unit variance then the output variance is given by $\|\phi\|_2$. The following problem is of relevance

$$\begin{aligned}\mu &= \min_{K \text{ stabilizing}} \|\phi(K)\|_2^2 \\ &= \min_{Q \text{ stable}} \{\|\phi(K)\|_2^2 : \phi = h - uq\}\end{aligned}$$

Suppose the unstable zeros of $u(\lambda)$ are given by $z_1, z_2, \ldots, z_n$ that are all distinct and real. Note that

$$
\begin{aligned}
& \Phi(\lambda) = H(\lambda) - U(\lambda)q(\lambda) \text{ for some } Q \text{ stable} \\
\Leftrightarrow \quad & \phi \text{ stable and } \tfrac{H(\lambda)-\Phi(\lambda)}{U(\lambda)} \text{ is stable} \\
\Leftrightarrow \quad & \Phi \text{ stable and } H(z_i) - \Phi(z_i) = 0 \text{ for all } i = 1, \ldots, n \\
\Leftrightarrow \quad & \Phi \text{ stable and } \textstyle\sum_{k=0}^{\infty} \phi(k) z_i^k = \sum_{k=0}^{\infty} h(k) z_i^k =: b_i \text{ for all } i = 1, \ldots, n \\
\Leftrightarrow \quad & \Phi \text{ stable and } \textstyle\sum_{k=0}^{\infty} \phi(k) z_i^k = \sum_{k=0}^{\infty} h(k) z_i^k =: b_i \text{ for all } i = 1, \ldots, n \\
\Leftrightarrow \quad & \phi \in \ell_1 \text{ and } \underbrace{\begin{pmatrix} 1 & z_1 & z_1^2 & \ldots \\ 1 & z_2 & z_2^2 & \ldots \\ \vdots & \vdots & \vdots & \\ 1 & z_n & z_n^2 & \ldots \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \vdots \end{pmatrix}}_{\phi} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{b} \\
\Leftrightarrow \quad & \phi \in \ell_1 \text{ and } A\phi = b.
\end{aligned}
$$

Thus the problem becomes

$$\mu = \inf_{\phi \in \ell_1, A\phi = b} \sum_{k=0}^{\infty} |\phi(k)|^2.$$

Using Lagrange multipliers the problem is equivalent to solving

$$\begin{array}{rcl} \mu = & & \max_{y \in R^n} \inf_{\phi \in \ell_1} \{\sum_{k=0}^{\infty} |\phi(k)|^2 + y^*[A\phi - b]\} \\ & = & \max_{y \in R^n} \inf_{\phi \in \ell_1} \{\sum_{k=0}^{\infty} |\phi(k)|^2 + \phi^* A^* y - b^* y]\} \end{array}$$

Note that

$$v := A^* y = \underbrace{\begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}}_{A^*} \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$$

Thus $v = \sum_{i=1}^{n} y_i v_i$. Thus

$$\begin{aligned} \mu &= \max_{y \in R^n} \inf_{\phi \in \ell_1} \{ \textstyle\sum_{k=0}^{\infty} |\phi(k)|^2 + \phi^* A^* y - b^* y] \} \\ &= \max_{y \in R^n} \inf_{\phi \in \ell_1} \{ \textstyle\sum_{k=0}^{\infty} |\phi(k)|^2 + \phi^* v - b^* y] \} \\ &= \max_{y \in R^n} \inf_{\phi \in \ell_1} \{ \textstyle\sum_{k=0}^{\infty} [|\phi(k)|^2 + \phi(k) v(k)] - b^* y \} \end{aligned}$$

Consider

$$\inf_{\phi \in \ell_1} \{ \sum_{k=0}^{\infty} [|\phi(k)|^2 + \phi(k) v(k)] - b^* y \}.$$

The solution can be obtained by minimizing over each $\phi(k)$ the expression

$$[|\phi(k)|^2 + \phi(k) v(k)]$$

which is minimized by

$$\phi(k) = -\frac{v(k)}{2}.$$

Thus

$$\begin{aligned}
\mu &= \max_{y\in R^n} \inf_{\phi\in\ell_1}\{\sum_{k=0}^{\infty}[|\phi(k)|^2 + \phi(k)v(k)] - b^*y\} \\
&= \max_{y\in R^n}\{\sum_{k=0}^{\infty}\tfrac{1}{4}v(k)^2 - \tfrac{1}{2}v(k)^2 - b^*y\} \\
&= \max_{y\in R^n}\{-\sum_{k=0}^{\infty}\tfrac{1}{4}v(k)^2 - b^*y\} \\
&= -\min_{y\in R^n}\{\sum_{k=0}^{\infty}\tfrac{1}{4}v(k)^2 + b^*y\}
\end{aligned}$$

Thus the problem reduces to solving the problem

$$\min_{y\in R^n} \frac{1}{4} y^*AA^*y + b^*y.$$

The solution to the above problem is given by

$$y = -2(AA^*)^{-1}b.$$

Thus the optimal v is given by

$$v^o = A^*y = -2A^*(AA^*)^{-1}b$$

and the optimal ϕ is given by

$$\phi = -\frac{1}{2}v = A^*(AA^*)^{-1}b$$

and the minimum value $\mu = (b^*(AA^*)^{-1}b)^{-1}$.