Design of Optimal Sparse Feedback Gains via the Alternating Direction Method of Multipliers (ADMM)

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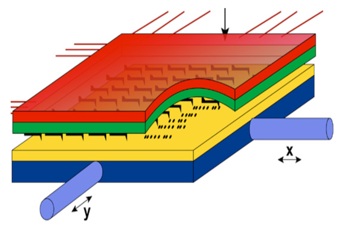
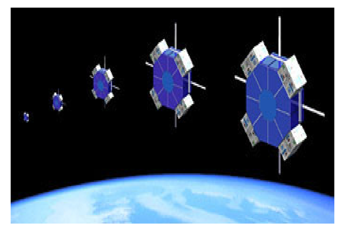
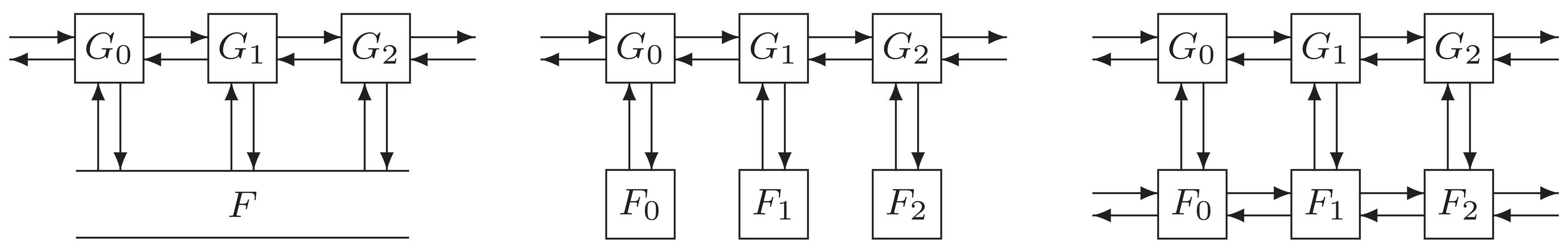
Large dynamic networks

- **Ubiquitous in modern technology**

wind farms	micro-cantilevers	satellite constellations
		

- **Controller architectures**



Centralized	Fully decentralized	Localized
best performance excessive communication	worst performance no communication	many possible architectures

Sparsity-promoting optimal control

$$\dot{x} = A\,x + B_1\,d + B_2\,u$$
$$z = \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix} \qquad u = -F\,x$$

Objective:

Strike a balance between variance amplification $d \to z$ and sparsity of F

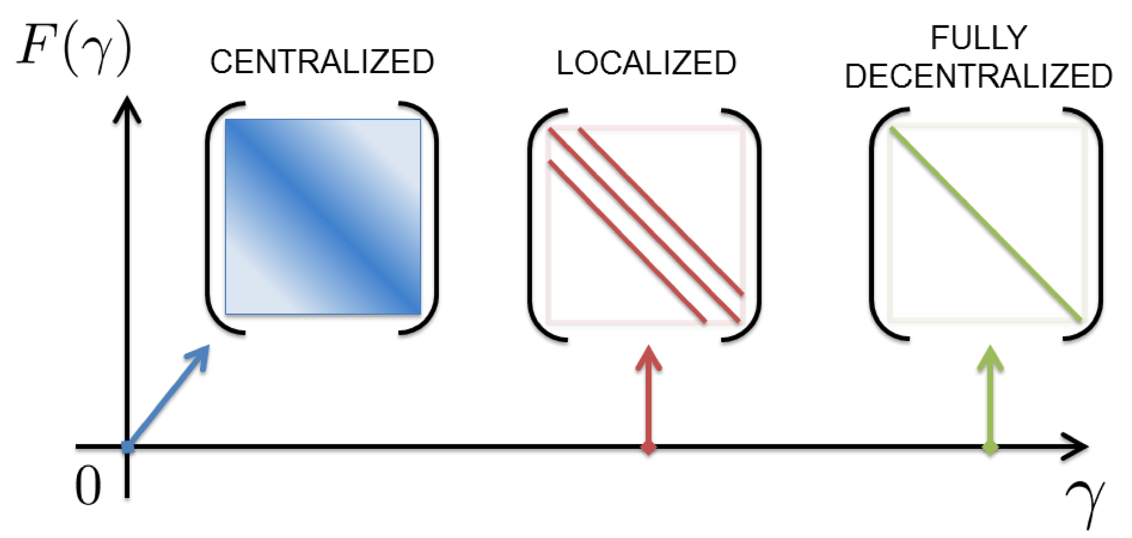
$$\text{minimize} \quad J(F) + \gamma\,\mathbf{card}\,(F)$$

Difficulties:

- Non-convex $J = \operatorname{trace}\left(\int_0^\infty \mathrm{e}^{(A-B_2F)^T t}\,(Q + F^T R F)\,\mathrm{e}^{(A-B_2F)t}\,\mathrm{d}t B_1 B_1^T\right)$
- Non-convex & discontinuous **card** = number of nonzero elements

Approaches:

- Identify convex problems/relaxations
- Utilize powerful distributed optimization algorithms
- Identify parameterized family of feedback gains



ADMM

- **Step 1: introduce additional variable/constraint**

$$\begin{array}{ll} \text{minimize} & J(F) + \gamma\,g(G) \\ \text{subject to} & F - G = 0 \end{array}$$

- **Step 2: introduce augmented Lagrangian**

$$\mathcal{L}_\rho = J(F) + \gamma\,g(G) + \operatorname{trace}\left(\Lambda^T(F - G)\right) + \frac{\rho}{2}\|F - G\|_F^2$$

- **Step 3: use ADMM for augmented Lagrangian minimization**

$$\begin{aligned} F^{k+1} &:= \arg\min_F \mathcal{L}_\rho(F, G^k, \Lambda^k) \\ G^{k+1} &:= \arg\min_G \mathcal{L}_\rho(F^{k+1}, G, \Lambda^k) \\ \Lambda^{k+1} &:= \Lambda^k + \rho(F^{k+1} - G^{k+1}) \end{aligned}$$

- **Step 4: Polishing – structured optimal design** $F \in \mathcal{S}$

Algorithm details

Sparsity-promoting penalty functions

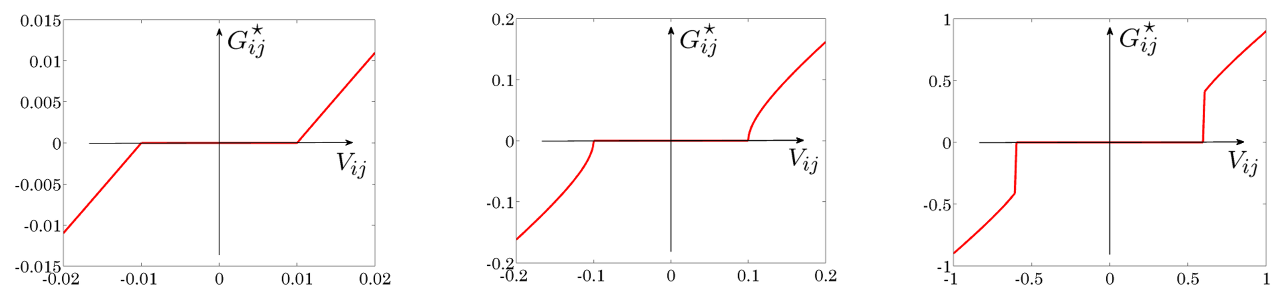
weighted ℓ_1	sum-of-logs	cardinality
$\sum_{i,j} W_{ij}\lvert F_{ij}\rvert$	$\sum_{i,j} \log\left(1 + \frac{\lvert F_{ij}\rvert}{\varepsilon}\right)$	$\sum_{i,j} \mathbf{card}(F_{ij})$

- **G-minimization step: Elementwise analytical solution**

weighted ℓ_1: **shrinkage** cardinality: **truncation**

$a = (\gamma/\rho)W_{ij}$ $\qquad$ $b = \sqrt{2\gamma/\rho}$

sum-of-log with $\{\rho = 100, \varepsilon = 0.1\}$



$\gamma = 0.1$ $\qquad$ $\gamma = 1$ $\qquad$ $\gamma = 10$

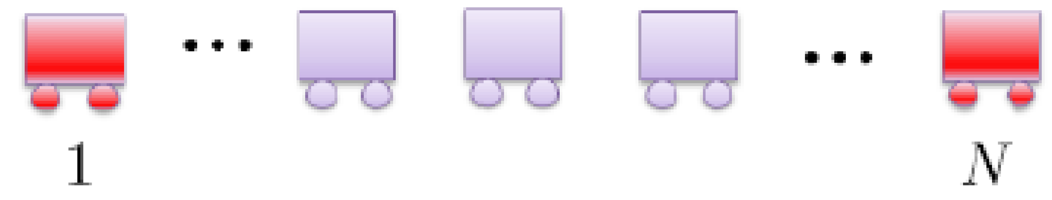
- **F-minimization step: Necessary conditions for optimality**

$$\begin{aligned} (A - B_2\,F)\,L + L\,(A - B_2\,F)^T &= -B_1\,B_1^T \\ (A - B_2\,F)^T\,P + P\,(A - B_2\,F) &= -(Q + F^T R\,F) \\ F\,L + \rho\,(2R)^{-1}\,F - R^{-1}B_2^T PL &= \rho\,(2R)^{-1}\,U \end{aligned}$$

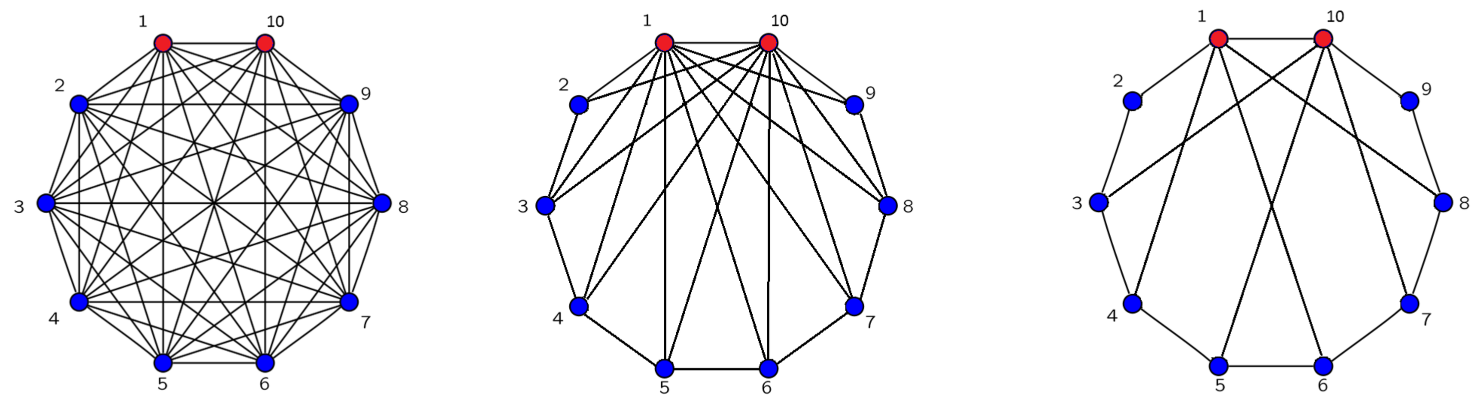
Descent scheme: $F_0 \to \{L_1, P_1\} \to F_1 \to \{L_2, P_2\} \cdots$

Examples

- **Formation of vehicles**



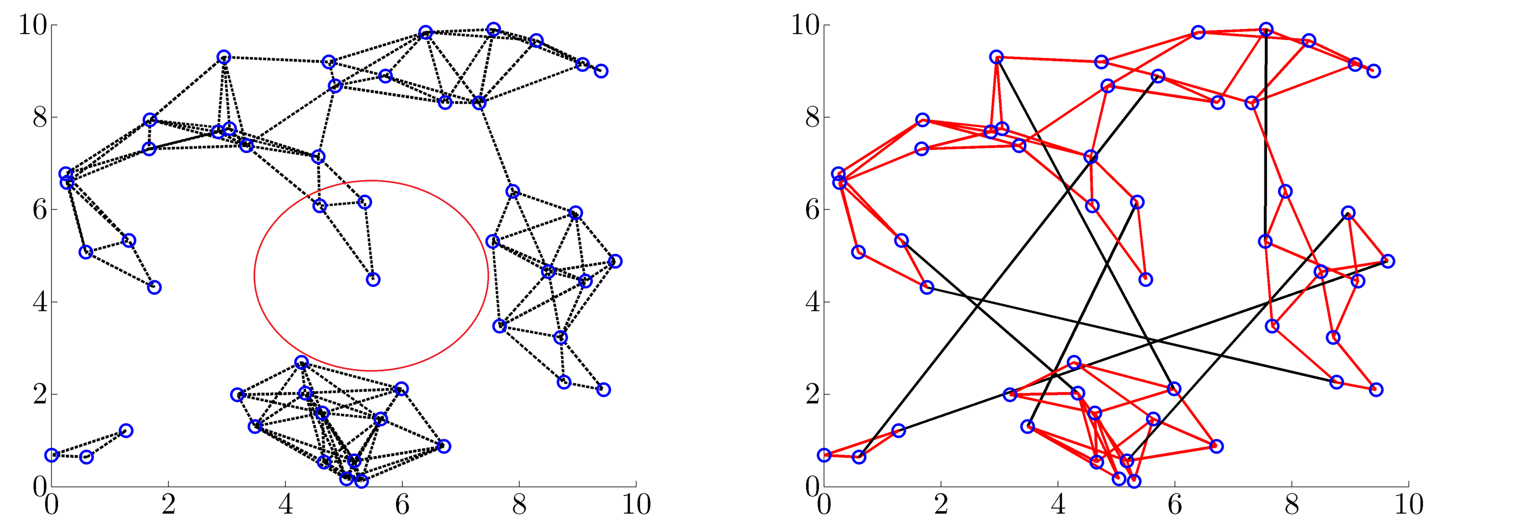
Identified communication architectures



$\gamma = 0$	$\gamma = 0.02$	$\gamma = 0.05$
all-to-all	nearest neighbors + leaders-to-all	nearest neighbors + leaders-to-some

- **Sensor network**

local performance graph: $\qquad$ **identified communication graph:**



$$z = \begin{bmatrix} Q_{\text{loc}}^{1/2}\,x \\ \left(I - \frac{1}{N}\mathbb{1}\mathbb{1}^T\right)x \\ u \end{bmatrix}$$

$$\begin{aligned} \mathbf{card}\,(F)/\mathbf{card}\,(F_c) &= 7\% \\ (J - J_c)/J_c &= 14\% \end{aligned}$$

Software

www.ece.umn.edu/~mihailo/software/lqrsp/

```
>> solpath = lqrsp(A,B1,B2,Q,R,options);
```

Publications

[1] F. Lin, M. Fardad, and M. R. Jovanović, "Design of optimal sparse feedback gains via the alternating direction method of multipliers", *IEEE Trans. Automat. Control*, submitted, 2012; also arXiv:1111.6188v2.

[2] F. Lin, M. Fardad, and M. R. Jovanović, "Optimal control of vehicular formations with nearest neighbor interactions", *IEEE Trans. Automat. Control*, doi:10.1109/TAC.2011.2181790, 2012.

[3] F. Lin, M. Fardad, and M. R. Jovanović, "Augmented Lagrangian approach to design of structured optimal state feedback gains", *IEEE Trans. Automat. Control*, vol. 56, no. 12, pp. 2923-2929, 2011.