Nonmodal amplification of disturbances as a route to elasticity-induced turbulence

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Nonmodal amplification of stochastic disturbances in elasticity-dominated channel flows of Oldroyd-B fluids is analyzed in this work. For streamwise-constant flows with high elasticity numbers $\mu$ and finite Reynolds numbers $Re$, we show that the linearized dynamics can be decomposed into slow and fast subsystems, and establish analytically that the steady-state variance amplification scales as $O(\mu Re^3)$. This demonstrates that large velocity variances can be sustained in weakly inertial/strongly elastic stochastically driven channel flows. The underlying physical mechanism involves polymer stretching that introduces an effective lift-up of flow fluctuations similar to vortex-tilting in inertial-dominated flows. The mechanism examined here provides a possible route for a bypass transition to elasticity-induced turbulence and might be exploited to enhance mixing in microfluidic devices.

1. Introduction

The classical approach to transition to turbulence examines the linearized governing equations for exponentially growing normal modes. The existence of these unstable modes implies exponential growth of infinitesimal perturbations to the laminar flow, and the corresponding eigenfunctions identify flow patterns that are expected to dominate early stages of transition. This approach agrees with experiments in many relevant flows (e.g., those driven by thermal and centrifugal forces (Trefethen et al. 1993)) but it comes up short in matching experimental observations in wall-bounded shear flows (flows in channels, pipes, and boundary layers). Failure of classical hydrodynamic stability analysis is attributed to the nonnormal nature of the linearized equations which may manifest itself by transient growth of perturbations (Butler & Farrell 1992), protrusion of pseudospectra to the unstable regions (Trefethen et al. 1993; Trefethen & Embree 2005), and large receptivity to ambient disturbances (Farrell & Ioannou 1993; Bamieh & Dahleh 2001; Jovanović & Bamieh 2005). Even in stable regimes—owing to nonnormality—perturbations that grow transiently before decaying due to viscosity can be configured, irregularities in laboratory design can lead to instability, and background disturbances (such as freestream turbulence or surface imperfections) can be amplified by orders of magnitude. These conclusions can be reached by performing transient growth, pseudospectra, or variance amplification analyses (Grossmann 2000; Schmid & Henningson 2001; Schmid 2007). All of these methods demonstrate the importance of streamwise-elongated flow patterns of high and low streamwise velocity in transitional wall-bounded shear flows of...
Newtonian fluids; this is at odds with classical modal stability results, but in agreement with experiments conducted in noisy environments (Matsubara & Alfredsson 2001).

Transition to turbulence in viscoelastic fluids is important from both fundamental and technological perspectives (Larson 1992). The observation that transition can occur even when the effects of fluid elasticity dominate those of fluid inertia—which is a primary cause of transition in Newtonian fluids—is particularly intriguing (Larson 2000; Groisman & Steinberg 2000, 2004; Berti et al. 2008). Improved understanding of transition mechanisms in viscoelastic fluids has broad applications, ranging from deeper insight into order-disorder transitions in spatially extended nonlinear dynamical systems to enhanced mixing in microfluidic devices through the addition of polymers (Groisman & Steinberg 2000, 2004; Berti et al. 2008). Amplification of stochastic disturbances in channel flows of viscoelastic fluids has recently been investigated using linear systems theory (Hoda, Jovanović & Kumar 2008). For the Oldroyd-B constitutive model—one of the simplest describing viscoelastic effects—computations reported in Hoda et al. (2008) demonstrated that streamwise-constant disturbances can be considerably amplified in the weakly inertial/strongly elastic regime. As in Newtonian fluids, this amplification is fundamentally nonmodal in nature: it cannot be described using the standard normal mode decomposition of classical hydrodynamic stability analysis (Grossmann 2000; Schmid & Henningson 2001; Schmid 2007). Rather, it arises due to an energy exchange involving the fluctuations in the streamwise/wall-normal polymer stress and the wall-normal gradient of the streamwise velocity (Hoda, Jovanović & Kumar 2009).

The key parameters in viscoelastic fluids are: the viscosity ratio, $\beta = \eta_s/ (\eta_s + \eta_p)$, where $\eta_s$ and $\eta_p$ are the solvent and polymer viscosities; the Reynolds number, $Re = \rho U_o L/(\eta_s + \eta_p)$, which represents the ratio of inertial to viscous forces; and the elasticity number, $\mu = We/Re$, which quantifies the strength of elastic forces relative to inertial forces. Here, $We = \lambda U_o / L$ is the Weissenberg number, $\lambda$ is the fluid relaxation time, $U_o$ is the largest base flow velocity, $L$ is the channel half-height, and $\rho$ is the fluid density. By modeling ambient disturbances as stochastic forcing to streamwise-constant channel flows of Oldroyd-B fluids (with spanwise wavenumber $k_z$), an explicit scaling of the variance (energy) amplification with the Reynolds number $Re$ was developed in Hoda et al. (2009),

$$E(k_z; Re, \beta, \mu) = f(k_z; \beta, \mu) Re + g(k_z; \beta, \mu) Re^3,$$

where $f$ and $g$ denote $Re$-independent functions. It is worth noting that $E$ quantifies the ensemble-average energy density (associated with the velocity field) of the statistical steady-state (Farrell & Ioannou 1993, 1994). In this work, considering the flows with $\mu \gg 1$, we apply singular perturbation techniques to establish that the steady-state variance scales as

$$E(k_z; Re, \beta, \mu) \approx \hat{f}(k_z; \beta) Re + \hat{g}(k_z; \beta) \mu Re^3.$$ 

This scaling was hypothesized by Hoda et al. (2009) on the basis of numerical data; here, we furnish an analytical proof of its validity.

The last expression should be compared to the expression for the variance amplification in Newtonian fluids (Bamieh & Dahleh 2001),

$$E_N(k_z; Re) = f_N(k_z) Re + g_N(k_z) Re^3.$$ 

At low $Re$ the $k_z$-dependence of $E_N$ is governed by $f_N(k_z)$, $E_N(k_z; Re) \approx f_N(k_z) Re$, and at high $Re$ it is governed by $g_N(k_z)$, $E_N(k_z) \approx Re^3 g_N(k_z)$ (Bamieh & Dahleh 2001). We note that the $k_z$-dependence of both $\hat{f}$ and $f_N$ is characterized by viscous dissipation, which clearly indicates that the behavior of Newtonian fluids with low $Re$ is dominated
by diffusion. On the other hand, the g-functions exhibit peaks at \( k_z \approx O(1) \); the values of \( k_z \) where these peaks take place identify the spanwise length scales of the most energetic response to stochastic forcing in Newtonian fluids with high \( Re \), and in Oldroyd-B fluids with high \( \mu Re^3 \). Our results thus show that—in viscoelastic fluids—even at arbitrarily low (but non-vanishing) Reynolds numbers the steady-state level of velocity variance may be dominated by the \( Re^3 \)-term owing to the elastic amplification of disturbances. This reveals the subtle interplay between inertial and elastic forces in Oldroyd-B fluids with low Reynolds/high elasticity numbers.

In addition to furnishing an analytical proof for the scaling of the variance amplification at high \( \mu \), the other major contribution of this paper is the discovery that the linearized dynamics can be decomposed into slow and fast subsystems. This observation is used to cast the equations into a standard singularly perturbed form for which existing methodology (Kokotović, Khalil & O’Reilly 1999) can be applied. The decomposition of the linearized dynamics at high \( \mu \) is not obvious a priori, and it takes advantage of the intrinsic time scale (\( \lambda \)) in the Oldroyd-B constitutive equation. In addition, it facilitates derivation of an explicit analytical expression for the steady-state velocity variance. Our success with uncovering the hitherto unknown dependence of the energy amplification on elasticity number points to the scaling and modeling steps as prerequisites for applying standard singular perturbation techniques.

Our subsequent developments are laid out next. In § 2, we describe the streamwise-constant linearized model and identify a slow-fast decomposition for different components of the frequency response operator. In § 3, we employ singular perturbation techniques to determine the scaling of the steady-state velocity variance with elasticity number in strongly elastic polymer solutions. In § 4, we provide an explicit analytical expression for the variance amplification and discuss physical mechanisms leading to amplification from different forcing to different velocity components. We also determine the spanwise length scales of flow structures that contribute most to the steady-state variance and show that these structures assume the form of high and low speed streaks. The major contributions are summarized in § 5 and the mathematical developments are relegated to the appendices.

2. Streamwise-constant linearized model

We consider incompressible channel flows of Oldroyd-B fluids with \( \epsilon = 1/\mu \ll 1 \). The equations governing the dynamics (up to first order) of velocity \( \mathbf{v} = [v_1 v_2 v_3]^T \), pressure \( p \), and polymer stress tensor \( \mathbf{\tau} \) fluctuations around base flow condition \( (\nabla, \mathbf{\tau}) \) are brought to a non-dimensional form by scaling length with \( L \), velocity with \( U_o \), polymer stresses with \( \eta_p U_o / L \), pressure with \( (\eta_s + \eta_p) U_o / L \), and time with \( \lambda \) (see Appendix A)

\[
\epsilon \mathbf{\dot{v}} = -Re (\nabla \mathbf{\dot{v}} + \nabla \mathbf{v} \cdot \mathbf{v}) - \nabla p + (1 - \beta) \nabla \cdot \mathbf{\tau} + \beta \nabla^2 \mathbf{v} + \sqrt{\epsilon Re} \mathbf{d},
\]

\[
0 = \nabla \cdot \mathbf{v},
\]

\[
\mathbf{\dot{\tau}} = \nabla \mathbf{v} + (\nabla \mathbf{v})^T - \mathbf{\tau} + \frac{Re}{\epsilon} (\mathbf{\tau} \cdot \nabla \mathbf{v} + \bar{\mathbf{\tau}} \cdot \nabla \mathbf{v} + (\bar{\mathbf{\tau}} \cdot \nabla \mathbf{v})^T + (\mathbf{\tau} \cdot \nabla \mathbf{v})^T - \nabla \mathbf{v} \cdot \mathbf{\bar{\tau}} - \nabla \mathbf{v} \cdot \mathbf{\tau}).
\]  

(2.1)

Here, a dot signifies a partial derivative with respect to time \( t \), \( \nabla \) is the gradient, \( \nabla \mathbf{v} = \mathbf{v} \cdot \nabla \), and \( v_1, v_2, \) and \( v_3 \) are the velocity fluctuations in the streamwise \( (x) \), wall-normal \( (y) \), and spanwise \( (z) \) directions, respectively. If \( t \) denotes time normalized by the convective time scale \( L/U_o \), then \( t = t/W_e \).

The linearized momentum equation is driven by the body force fluctuation vector \( \mathbf{d} = [d_1 d_2 d_3]^T \), which is considered to be purely harmonic in the horizontal directions,
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and stochastic in the wall-normal direction and in time. This spatio-temporal forcing will in turn yield velocity and polymer stress fluctuations of the same nature. Our goal is to study the steady-state level of variance maintained in $v$ by assuming that $d$ is a temporally stationary white Gaussian process with zero mean and unit variance. It is noteworthy that if $\tilde{d}$ denotes the forcing in the standard non-dimensional momentum equation (with time scale $L/U_o$), then $d(r,t) = \sqrt{We} \tilde{d}(r,\tilde{t})$, where $r$ is the position vector, $r = [x \ y \ z]^T$. This body-force scaling is introduced to guarantee the same auto-correlation operators of $d(r,t)$ and $\tilde{d}(r,\tilde{t})$ (Kokotović et al. 1999).

2.1. Frequency response representation

We study the linearized model for streamwise-constant three-dimensional fluctuations, which means that the dynamics evolve in the $(y,z)$-plane, but flow fluctuations in three spatial directions are considered. This model is analyzed since the largest variance in stochastically forced channel flows of viscoelastic fluids is maintained by streamwise-constant fluctuations (Hoda et al. 2008). The linearized equations can be brought to an evolution form by removing the pressure from the equations and by expressing $v$ in terms of the wall-normal velocity/vorticity ($v_2, \omega_2$) fluctuations (see Appendix B). Furthermore, application of the Fourier transform in $t$ and $z$ allows for elimination of the polymer stresses from the evolution model, which results in an equivalent frequency response representation of the linearized system (see Appendix C). The block diagram in figure 1 provides a systems-level view of the streamwise-constant model. The boxes represent different parts of the system and the circles denote summation of signals. Inputs into each box/circle are represented by lines with arrows directed toward the box/circle, and outputs of each box/circle are represented by lines with arrows leading away from the box/circle. The inputs specify the signals affecting subsystems, and the outputs designate the signals of interest or signals affecting other parts of the system (Åström & Murray 2008).

All signals in figure 1 are functions of the wall-normal coordinate $y$, the spanwise wave-number $k_z$, and the temporal frequency $\omega$, e.g. $v_2 = v_2(y, k_z, \omega)$, with the following boundary conditions on $v_2$ and $\omega_2$, $\{v_2(\pm1, k_z, \omega) = \partial_y v(\pm1, k_z, \omega) = \omega_2(\pm1, k_z, \omega) = 0\}$. The capital letters in figure 1 denote the Reynolds-number-independent operators. These operators act in the wall-normal direction and some of them are parameterized by $k_z$ ($F_j$ and $G_j$, $j = 1, 2, 3$), while the others depend on $k_z$, $\omega$, $\beta$, and $\epsilon$ ($J_1$, $J_2$, and $C_p$). The operators $F_j$ and $G_j$ are given by

\[
F_1 = ik_z, \quad F_2 = -k_z^2 \Delta^{-1}, \quad F_3 = -ik_z \Delta^{-1} \partial_y, \quad G_1 = -(i/k_z), \quad G_2 = I, \quad G_3 = (i/k_z) \partial_y,
\]

and they, respectively, describe the way the forcing enters into the evolution model, and

Figure 1. Block diagram of the linearized streamwise-constant model.
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the way the velocity fluctuations depend on the wall-normal velocity and vorticity. The operator $C_p$ captures the coupling from $v_2$ to $\omega_2$, and it is defined as

$$C_p = C_{p1} + \frac{1}{\epsilon(1 + i\omega)^2} C_{p2},$$

where $C_{p1} = -ik_z U'(y)$ denotes the vortex tilting term (Butler & Farrell 1992), and

$$C_{p2} = ik_z (1 - \beta) C_{p2}, \quad \bar{C}_{p2} = U'(y) \Delta + 2U''(y) \partial_y,$$

denotes the term arising due to the work done by the polymer stresses on the flow (Hoda et al. 2009). Finally, $J_1$ and $J_2$ govern the internal dynamics of the wall-normal vorticity and velocity fluctuations, respectively. These two operators are given by

$$J_j = (1 + i\omega) K_j, \quad K_j = \left( (\epsilon i \omega)^2 I - (i \beta T_j - \epsilon I) i \omega - T_j \right)^{-1}.$$

Here, $T_1 = \Delta$ and $T_2 = \Delta^{-1} \Delta^2$, respectively, stand for the Squire and Orr-Sommerfeld operators in the streamwise-constant model of Newtonian fluids with $Re = 1$ (Schmid & Henningson 2001). $I$ is the identity operator, $\Delta = \partial_{yy} - k_z^2$ is a Laplacian with homogeneous Dirichlet boundary conditions, $\Delta^{-1}$ is the inverse of the Laplacian, $\Delta^2 = \partial_{yy} - 2k_z^2 \partial_{yy} + k_z^4$ with homogeneous Cauchy (both Dirichlet and Neumann) boundary conditions, $i = \sqrt{-1}$. $U(y) = y$ in Couette flow, $U(y) = 1 - y^2$ in Poiseuille flow, and $U'(y) = dU(y)/dy$.

In the frequency domain, the velocity and forcing components are related by $v_i = H_{ij} d_j$, where $H_{ij}$ denotes the $ij$th component of the frequency response operator $H$,

$$v(y, k_z, \omega; Re, \beta, \epsilon) = [H(k_z, \omega; Re, \beta, \epsilon) d(\omega)] (y).$$

The variance maintained in $v$ is given by (Farrell & Ioannou 1994)

$$E(k_z; Re, \beta, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( H(k_z, \omega; Re, \beta, \epsilon) H^*(k_z, \omega; Re, \beta, \epsilon) \right) d\omega,$$

where $H^*$ is the adjoint of operator $H$. Using the properties of the trace operator we have $E = \sum_{i,j=1}^{3} E_{ij}$, where $E_{ij}$ is the variance maintained in $v_i$ by stochastically forcing the linearized model with $d_j$. From the definitions of $E$ and operators $H_{ij}$, the following $Re$-scaling of variance amplification was recently obtained in Hoda et al. (2009)

$$E(k_z; Re, \beta, \epsilon) = f(k_z; \beta, \epsilon) Re + g(k_z; \beta, \epsilon) Re^3. \quad (E)$$

This expression for the steady-state variance of $v$ is valid for all $k_z$, $Re$, $\beta$, and $\epsilon$, and it follows from the fact that the $H_{ij}$ are determined by (see Appendix C)

$$H_{1j}(k_z, \omega; Re, \beta, \epsilon) = \sqrt{Re} \bar{H}_{1j}(k_z, \omega; \beta, \epsilon), \quad j = 2, 3,$$
$$H_{ij}(k_z, \omega; Re, \beta, \epsilon) = \sqrt{Re} \bar{H}_{ij}(k_z, \omega; \beta, \epsilon), \quad i, j = 2, 3; \quad i = j = 1,$$
$$H_{11}(k_z, \omega; Re, \beta, \epsilon) = 0, \quad i = 2, 3,$$

where the $\bar{H}_{ij}$ represent the $Re$-independent operators,

$$\bar{H}_{11} = \sqrt{\epsilon}(1 + i\omega) G_1 K_1 F_1, \quad \bar{H}_{ij} = \sqrt{\epsilon}(1 + i\omega) G_i K_2 F_j, \quad i, j = 2, 3,$$
$$\bar{H}_{1j} = \frac{1}{\sqrt{\epsilon}} G_1 K_1 (\epsilon(1 + i\omega)^2 C_{p1} + C_{p2}) K_2 F_j, \quad j = 2, 3.$$

Furthermore,

$$f = f_{11} + f_{22} + f_{23} + f_{32} + f_{33}, \quad g = g_{12} + g_{13},$$
Evolution equations for \( \bar{x} \) auto-correlation operator of \( \bar{\sigma} \) obtain the following evolution equation different for each \( \bar{f} \) and similarly for \( \bar{v} \) response operator \( \bar{d} \) determined by recasting each \( \bar{d} \). The Reynolds-number-independent contributions to the steady-state variance can be mit a standard singularly perturbed form which is convenient for uncovering dependence of the variance amplification on elasticity number.

2.2. Evolution equations for \( \bar{H}_{ij} \)

The Reynolds-number-independent contributions to the steady-state variance can be determined by recasting each \( \bar{H}_{ij} \) in the evolution form

\[
\begin{align*}
\dot{x}_{ij}(y, k_z, t) &= A_{ij}(k_z) x_{ij}(y, k_z, t) + B_{ij}(k_z) d_j(y, k_z, t), \\
\dot{v}_i(y, k_z, t) &= C_i(k_z) x_{ij}(y, k_z, t),
\end{align*}
\]

where \( x_{ij} \) is a vector of state variables, and \((d_j, v_i)\) is the input-output pair for frequency response operator \( \bar{H}_{ij} \). Note that \( x_{ij} \), and operators \( A_{ij}, B_{ij}, \) and \( C_i \) will, in general, be different for each \( \bar{H}_{ij} \). It is a standard fact (Farrell & Ioannou 1993) that the variance of \( v_i \) sustained by \( d_j \) is determined by trace \( (P_{ij} C_i^T C_i) \), where \( P_{ij} \) denotes the steady-state auto-correlation operator of \( x_{ij} \), which is found by solving the Lyapunov equation,

\[
A_{ij} P_{ij} + P_{ij} A_{ij}^* = -B_j B_j^*.
\]

**Evolution equations for \( \bar{H}_{ij} \) with \( i = j = 1 \) and \( i, j = 2,3 \)**

The operators \( \{\bar{H}_{11} = \sqrt{\epsilon}(1 + i \omega) G_1 K_1 F_1; \bar{H}_{ij} = \sqrt{\epsilon}(1 + i \omega) G_i K_j F_j, i,j = 2,3\} \) can be represented by the following set of equations (with the left-hand side denoting relations in the frequency domain, and the right-hand side denoting relations in the time domain):

\[
\begin{align*}
\sigma_{kj} &= k_i F_j d_j \\
v_i &= \sqrt{\epsilon}(1 + i \omega) G_i \sigma_{kj} \Rightarrow v_i &= \sqrt{\epsilon} G_i \sigma_{kj},
\end{align*}
\]

with \( \{k = 1 \text{ for } j = 1; k = 2 \text{ for } j = 2,3\} \). By selecting \( x_{kj} = \sigma_{kj} \) and \( z_{kj} = \hat{\sigma}_{kj} \) we obtain the following evolution equation

\[
\begin{align*}
\begin{bmatrix} \dot{x}_{ij} \\ \epsilon \dot{z}_{ij} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ T_k & \beta T_k - \epsilon I \end{bmatrix} \begin{bmatrix} x_{ij} \\ z_{ij} \end{bmatrix} + \begin{bmatrix} 0 \\ F_j \end{bmatrix} d_j, \\
\dot{v}_i &= \sqrt{\epsilon} \begin{bmatrix} G_i \\ G_i \end{bmatrix} \begin{bmatrix} x_{ij} \\ z_{ij} \end{bmatrix},
\end{align*}
\]

with \( \{k = 1 \text{ for } i = 1; k = 2 \text{ for } i = 2,3\} \), homogeneous Dirichlet boundary conditions on \( x_{11} \) and \( z_{11} \), and homogeneous Cauchy boundary conditions on \( x_{ij} \) and \( z_{ij} \) for \( i,j = 2,3 \).

**Evolution equations for \( \bar{H}_{ij} \) with \( j = 2,3 \)**

From Appendix C it follows that

\[
\bar{H}_{ij} = (1/\sqrt{\epsilon}) G_i K_1 (\epsilon(1 + i \omega)^2 C_{p1} + C_{p2}) K_2 F_j, j = 2,3,
\]

which yields an equivalent block diagram representation of \( \bar{H}_{12} \) and \( \bar{H}_{13} \) in figure 2.
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\[ \psi = K_2 F_j d_j \implies \epsilon \dot{\psi} = T_2 \psi + (\beta T_2 - \epsilon I) \psi + F_j d_j, \]
\[ \varphi = (\epsilon (1 + i \omega)^2 C_{p1} + C_{p2}) \psi \implies \varphi = C_{p1}(\epsilon \psi^2 + 2 \epsilon \dot{\psi} + \psi) + C_{p2} \psi, \]
\[ \phi = K_1 \varphi \implies \epsilon \dot{\phi} = T_1 \phi + (\beta T_1 - \epsilon I) \phi + \varphi, \]
\[ v_1 = (1/\sqrt{\epsilon}) G_1 \phi \implies v_1 = (1/\sqrt{\epsilon}) G_1 \phi, \]

with homogeneous Dirichlet boundary conditions on \( \phi \), and homogeneous Cauchy boundary conditions on \( \psi \). By selecting \( x = [\psi \ \phi]^T \), \( z = [\dot{\psi} \ \dot{\phi}]^T \), we obtain a singularly perturbed realization of \( \hat{H}_{ij}, j = 2, 3 \),

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
0 & I \\
A_{21}(\epsilon) & A_{22}(\epsilon)
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
0 \\
B_2
\end{bmatrix} d_j,
\]

\[ v_1 = \sqrt{1/\epsilon} \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \quad (2.3) \]

where all operators are partitioned conformably with the elements of \( x \) and \( z \),

\[
A_{21}(\epsilon) = \begin{bmatrix}
T_2 & 0 \\
C_{p2} + C_{p1}(T_2 + \epsilon I) & T_1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
F_j \\
C_{p1} F_j
\end{bmatrix},
\]
\[
A_{22}(\epsilon) = \begin{bmatrix}
\beta T_2 - \epsilon I & 0 \\
C_{p1}(\beta T_2 + \epsilon I) & \beta T_1 - \epsilon I
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
0 & G_1
\end{bmatrix}.
\]

Equations (2.2) and (2.3) are in the standard singularly perturbed form (Kokotović et al. 1999) as the time-derivative of the second part of the state is multiplied by a small positive parameter \( \epsilon \) and the lower-right-hand-corner-blocks of dynamical generators in both (2.2) and (2.3) at \( \epsilon = 0 \) are invertible. Furthermore, this representation gives evolution equations for different components of the frequency response operator with a lower number of states compared to the original evolution model (B1). In particular, there are two state variables in the evolution equations for operators \( \hat{H}_{ij} \) with \( \{i = j = 1; i, j = 2, 3\} \) (cf. (2.2)), and four state variables in the evolution equations for operators \( \hat{H}_{12} \) and \( \hat{H}_{13} \) (cf. (2.3)). (In comparison, there are eight states in the evolution model (B1).) We exploit the structure of equations (2.2) and (2.3) in § 3 to uncover a slow-fast decomposition of each \( \hat{H}_{ij} \) and provide an explicit analytical expression for the steady-state velocity variance in flows of strongly elastic polymer solutions. These analytical developments are then utilized in § 4 to clearly identify important physical mechanisms leading to amplification from different forcing to different velocity components.

3. Singular perturbation analysis of variance amplification

Considering the case of high \( \mu, \epsilon = 1/\mu \ll 1 \), we now apply singular perturbation methods to show how functions \( f \) and \( g \) in the expression for the steady-state velocity
variance (\(E\)) scale with the elasticity number. From § 2.2 it follows that the evolution equations of operators \(H_{ij}\) assume the form

\[
\begin{bmatrix}
\dot{x} \\
\epsilon \dot{z}
\end{bmatrix} = \begin{bmatrix}
0 & I \\
A_{21}(\epsilon) & A_{22}(\epsilon)
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} + \begin{bmatrix}
0 \\
B_2
\end{bmatrix} d_j,
\]

(3.1)

with appropriate boundary conditions on \(x\) and \(z\), \(r(\epsilon) = \sqrt{\epsilon}\) for \(\{i = j = 1; i, j = 2, 3\}\), and \(r(\epsilon) = 1/\sqrt{\epsilon}\) for \(\{i = 1; j = 2, 3\}\). To simplify notation we have omitted \(i\) and \(j\) indices in (3.1); it is to be noted, however, that \(x, z, r,\) and the \(A\)-operators depend on both \(i\) and \(j\), the \(B\)-operators depend on \(j\), and the \(C\)-operators depend on \(i\). The following coordinate transformation (Kokotović et al. 1999)

\[
\begin{bmatrix}
\xi \\
\eta
\end{bmatrix} = \begin{bmatrix}
1 - \epsilon Q(\epsilon)L(\epsilon) & -\epsilon Q(\epsilon) \\
L(\epsilon) & I
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix},
\]

(3.2)

can be utilized to fully separate the slow and fast dynamics of system (3.1). Namely, if operators \(L(\epsilon)\) and \(Q(\epsilon)\) satisfy

\[
\begin{align*}
A_{21}(\epsilon) - A_{22}(\epsilon)L(\epsilon) - \epsilon L(\epsilon)L(\epsilon) &= 0, \\
I - Q(\epsilon)(A_{22}(\epsilon) + \epsilon L(\epsilon)) - \epsilon L(\epsilon)Q(\epsilon) &= 0,
\end{align*}
\]

(L) \quad (Q)

then the change of coordinates (3.2) brings system (3.1) to the following equivalent representation

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\zeta}
\end{bmatrix} = \begin{bmatrix}
A_s(\epsilon) & 0 \\
0 & \frac{1}{\epsilon}A_f(\epsilon)
\end{bmatrix}
\begin{bmatrix}
\xi \\
\zeta
\end{bmatrix} + \begin{bmatrix}
\frac{1}{\epsilon}B_s(\epsilon) \\
B_f(\epsilon)
\end{bmatrix} d_j,
\]

(3.4)

with

\[
\begin{align*}
A_s(\epsilon) &= -L(\epsilon), & A_f(\epsilon) &= A_{22}(\epsilon) + \epsilon L(\epsilon), \\
B_s(\epsilon) &= -Q(\epsilon)B_2, & B_f(\epsilon) &= B_2, \\
C_s(\epsilon) &= C_1 - C_2L(\epsilon), & C_f(\epsilon) &= C_2 + \epsilon(C_1 - C_2L(\epsilon))Q(\epsilon).
\end{align*}
\]

(3.5)

The \(\xi\)- and \(\zeta\)-subsystems in (3.4) identify the slow and fast subsystems, respectively. It is noteworthy that \(\zeta\) penalizes the deviation of \(z\) from the slow invariant manifold \(\bar{z} = -L(\epsilon)x\), i.e. \(\zeta = z - \bar{z} = z + L(\epsilon)x\). (Note that \(\bar{z} = -L(\epsilon)x\) is an invariant manifold of unforced system (3.1) if \(\bar{z}(t_0, \epsilon) = -L(\epsilon)x(t_0, \epsilon)\) implies \(\bar{z}(t, \epsilon) = -L(\epsilon)x(t, \epsilon)\) for every \(t \geq t_0\). On the other hand, the variable \(\xi\) is introduced to fully decouple the influence of \(\zeta\) on the slow subsystem. We refer the reader to Kokotović et al. (1999) for additional details on slow-fast decomposition of singularly perturbed systems.

It is now easy to show that the steady-state auto-correlation operator of \(\left[ \begin{array}{c} \xi^T \\ \zeta^T \end{array} \right]^T\) takes the form

\[
\bar{P}(\epsilon) = \begin{bmatrix}
X(\epsilon) & Y^*(\epsilon) \\
Y(\epsilon) & (1/\epsilon)Z(\epsilon)
\end{bmatrix},
\]

where components of \(\bar{P}\) are to be determined from the following system of equations

\[
\begin{align*}
A_s(\epsilon)X(\epsilon) + X(\epsilon)A_s^*(\epsilon) &= -B_s(\epsilon)B_s^*(\epsilon), \\
A_f(\epsilon)Y(\epsilon) + \epsilon Y(\epsilon)A_f^*(\epsilon) &= -B_f(\epsilon)B_f^*(\epsilon), \\
A_f(\epsilon)Z(\epsilon) + Z(\epsilon)A_f^*(\epsilon) &= -B_f(\epsilon)B_f^*(\epsilon),
\end{align*}
\]

(3.6)
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This implies that the steady-state variance of operator \( \bar{H}_{ij} \) (cf. § 2.1) is given by

\[
E_{ij} = r^2(\epsilon) \text{trace}(X(\epsilon)C_s^*(\epsilon)C_s(\epsilon) + (1/\epsilon)Z(\epsilon)C_f^*(\epsilon)C_f(\epsilon)) + \text{trace}(Y(\epsilon)C_s^*(\epsilon)C_f(\epsilon) + C_f^*(\epsilon)C_s(\epsilon)Y^*(\epsilon)).
\]

(3.7)

Now, using the fact that \( A_{21}(\epsilon) \) and \( A_{22}(\epsilon) \) in equations (2.2) and (2.3) are given by

\[
A_{21}(\epsilon) = A_{21,0} + \epsilon A_{21,1}, \quad A_{22}(\epsilon) = A_{22,0} + \epsilon A_{22,1},
\]

we represent \( L \) and \( Q \) as

\[
L(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n L_n, \quad Q(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n Q_n,
\]

and employ (regular) perturbation analysis to render \((L)\) and \((Q)\) into the following set of conveniently coupled equations:

\[
\begin{align*}
\epsilon^0 : & \quad \left\{ \begin{array}{l}
L_0 = A_{22,0}^{-1} A_{21,0}, \\
Q_0 = A_{22,0}^{-1},
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\epsilon^1 : & \quad \left\{ \begin{array}{l}
L_1 = A_{22,0}^{-1} (A_{21,1} - A_{22,1}L_0 - L_0L_0), \\
Q_1 = -(L_0Q_0 + Q_0 (A_{22,1} + L_0)) A_{22,0}^{-1},
\end{array} \right.
\end{align*}
\]

Based on (3.5) we conclude that \( \Pi(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Pi_n \) where \( \Pi \) stands for \( A_s, A_f, B_s, C_s, \) or \( C_f; \) this implies that a similar procedure can be used to simplify (3.6) and determine coefficients in the power series expansions of operators \( X, Y, \) and \( Z \)

\[
\begin{align*}
\epsilon^0 : & \quad \left\{ \begin{array}{l}
A_{s,0}X_0 + X_0A_{s,0}^* = -B_{s,0}B_{s,0}^*, \\
A_{f,0}Y_0 = -B_fB_f^*,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\epsilon^1 : & \quad \left\{ \begin{array}{l}
A_{s,0}X_1 + X_1A_{s,0}^* = -(A_{s,1}X_0 + X_0A_{s,1}) - (B_{s,0}B_{s,1}^* + B_{s,1}B_{s,0}^*), \\
A_{f,0}Y_1 + B_fB_{s,1}^* = -(A_{f,1}Y_0 + Y_0A_{s,0}^*),
\end{array} \right.
\end{align*}
\]

The above results are used in what follows to reveal the scaling of functions \( f \) and \( g \) in (E) with elasticity number.

3.1. Scaling of functions \( f_{ij} \) with \( \mu \)

We next examine how functions \( f_{ij}(k_z; \beta, \epsilon) \) depend on \( \epsilon \) in flows with \( \epsilon = 1/\mu \ll 1. \) The \( Re \)-independent operators

\[
\bar{H}_{11} = \sqrt{\epsilon} (1 + i\omega) G_1K_1F_1, \quad \bar{H}_{ij} = \sqrt{\epsilon} (1 + i\omega) G_iK_jF_j, \quad i, j = 2, 3,
\]

admit the evolution equations given by (2.2). As shown in § 2.1, \( f(k_z; \beta, \epsilon) = \sum_{i,j} f_{ij}(k_z; \beta, \epsilon), \) \( \{i = j = 1; i, j = 2, 3\}, \) where \( f_{ij} \) denotes the steady-state variance of system (2.2). A direct comparison of equations (2.2) and (3.1) yields

\[
r(\epsilon) = \sqrt{\epsilon}, \quad A_{21} = T_k, \quad A_{22} = \beta T_k - \epsilon I, \quad B_2 = F_j, \quad C_1 = C_2 = G_i,
\]
which in combination with (3.7) and the above perturbation analysis can be used to obtain (see Appendix D.1)

\[ f(k_z; \beta, \epsilon) = \hat{f}_0(k_z; \beta) + \sum_{n=1}^{\infty} \epsilon^n \hat{f}_n(k_z; \beta). \]

Here, \( \hat{f}_n \) are functions independent of \( \epsilon \). Thus, in streamwise-constant channel flows with \( \epsilon = 1/\mu \ll 1 \), the functions contributing to the \( Re \)-scaling of the steady-state velocity variance in (E) approximately become \( \mu \)-independent, i.e.

\[ f(k_z; \beta, \epsilon) = \hat{f}_0(k_z; \beta) + O(\epsilon), \]

where \( \hat{f}_0(k_z; \beta) \) is determined by the terms of the form \( \text{trace}(Z_0 C_{f,0}^* C_f) \), with \( A_{f,0} Z_0 + Z_0 A_{f,0}^* = -B_f B_f^* \); see Appendix D.1 for details.

### 3.2. Scaling of functions \( g_{1j} \) with \( \mu \)

By examining \( \hat{H}_{12} \) and \( \hat{H}_{13} \) we can determine the \( \mu \)-dependence of terms responsible for the \( Re^3 \)-scaling of the steady-state variance. As shown in § 2.1, \( g(k_z; \beta, \epsilon) = \sum_{j=2}^{\infty} g_{1j}(k_z; \beta, \epsilon) \), where \( g_{1j} \) denotes the steady-state variance of system (2.3). By comparing equations (2.3) and (3.1) we see that \( r(\epsilon) = 1/\sqrt{7} \) and \( C_2 \equiv 0 \). The latter observation implies that \( C_s = C_1 = \begin{bmatrix} 0 & \mathbf{G}_1 \end{bmatrix} \) is \( \epsilon \)-independent and that \( C_f(\epsilon) = \epsilon C_1 \mathbf{Q}(\epsilon) \), which together with (3.7) can be used to obtain

\[ g_{1j} = (1/\epsilon) \text{trace}(X_{j,0} C_j^* C_1) + O(1). \]

In fact, a closer examination of (3.7) in conjunction with the above perturbation analysis shows that (see Appendix D.2)

\[ g(k_z; \beta, \epsilon) = \frac{1}{\epsilon} \sum_{n=0}^{\infty} \epsilon^n \hat{g}_n(k_z; \beta), \quad \epsilon \ll 1, \]

where \( \hat{g}_0(k_z; \beta) = \sum_{j=2,3} \text{trace}(X_{j,0} C_j^* C_1) \), with \( A_{s,0} X_{j,0} + X_{j,0} A_{s,0}^* = -B_{s,0} B_{s,0}^* \).

Hence, in streamwise-constant channel flows with \( \epsilon = 1/\mu \ll 1 \), the functions contributing to the \( Re^3 \)-scaling of the steady-state velocity variance approximately scale linearly with \( \mu \).

### 4. Variance amplification in elasticity-dominated flows

Based on the above developments, it follows that in streamwise-constant Poiseuille and Couette flows of Oldroyd-B fluids with sufficiently large \( \mu \), the variance maintained in \( \mathbf{v} \) is given by

\[ E(k_z; R e, \beta, \mu) \approx R e \hat{f}_0(k_z; \beta) + \mu R e^3 \hat{g}_0(k_z; \beta), \]

where \( \hat{f}_0 \) and \( \hat{g}_0 \) are functions independent of \( Re \) and \( \mu \). Therefore, in elasticity-dominated flows, the terms responsible for the \( Re \) - and \( Re^3 \)-scaling of the steady-state variance are, respectively, \( \mu \)-independent and linearly dependent on \( \mu \). As noted in § 1, this \( \mu \)-scaling was suggested by numerical data from the frequency response analysis of Hoda et al. (2009). Here, we have shown how it follows directly from the governing equations by decomposing the linearized dynamics into slow and fast subsystems.

It is readily shown that

\[ \hat{f}_0(k_z; \beta) = \hat{f}_0(k_z)/\beta, \]
where the base-flow-independent function \( \tilde{f}_0(k_z) \) is given by

\[
\tilde{f}_0(k_z) = f_N(k_z) = -0.5 \left( \text{trace} \left( T_1^{-1} \right) + \text{trace} \left( T_2^{-1} \right) \right),
\]

with \( f_N(k_z) \) being the function that arises in the expression for variance amplification in Newtonian fluids as noted in § 1. Furthermore,

\[
\tilde{g}_0(k_z; \beta) = \tilde{g}_0(k_z)(1 - \beta)^2 / \beta,
\]

with

\[
\tilde{g}_0(k_z) = (k_z^2 / 4) \text{trace} \left( T_1^{-1} \hat{C}_{p2} T_2^{-2} \hat{C}_{p2}^* T_1^{-1} \right).
\]

Thus, for \( \mu \gg 1 \),

\[
E(k_z; Re, \beta, M) \approx \frac{Re}{\beta} \left( \tilde{f}_0(k_z) + M^2(1 - \beta)^2 \tilde{g}_0(k_z) \right),
\]

where \( M \) denotes the viscoelastic Mach number (Joseph 2001), \( M^2 = \mu Re c^2 = We Re = (U_\infty / c)^2 \), and \( c \) is the shear-wave speed, \( c^2 = (\eta_s + \eta_p) / (\rho \lambda) \). This shows that the variance depends affinely on \( M^2 \) and that it increases monotonically with a decrease in the ratio of the solvent viscosity to the total viscosity.

The analytical expressions for trace \( (T_j)^{-1} \), \( j = 1, 2 \), were derived in Bamieh & Dahleh (2001); these can be used to explicitly calculate \( \tilde{f}_0(k_z) = f_N(k_z) \), which is illustrated in figure 3(a). In Couette flow, the expression for \( \tilde{g}_0(k_z) \) simplifies to \(- (k_z^2 / 4) \text{trace} \left( T_2^{-2} T_1^{-1} \right)\), and an explicit \( k_z \)-dependence of \( \tilde{g}_0 \) can be derived after some manipulation. The resulting expression for \( \tilde{g}_0(k_z) \) is used to generate the plot in figure 3(b); from this plot we observe the non-monotonic character of \( \tilde{g}_0(k_z) \), with peak values at \( k_z \approx 2.07 \) (in Couette flow) and \( k_z \approx 2.24 \) (in Poiseuille flow). (In Poiseuille flow, determination of the expression for \( \tilde{g}_0(k_z) \) is considerably more involved than in Couette flow; however, the method developed by Jovanović & Bamieh (2006) can be used to compute this quantity efficiently without resorting to spatial discretization of the underlying operators.). Plots illustrating influence of the Mach number on the \( k_z \)-dependence of \( \tilde{f}_0(k_z) + M^2(1 - \beta)^2 \tilde{g}_0(k_z) \), in Poiseuille flow with \( \beta = 0.1 \), are shown in figure 3(c). Clearly, the behavior at low \( M \) is governed by viscous dissipation, \( \tilde{f}_0(k_z) \); as \( M \) increases, additional physical effects become important which leads to appearance of the peaks at \( k_z \approx O(1) \). As mentioned earlier, the values of \( k_z \) where these peaks emerge determine the spanwise length scales of the most energetic response to stochastic forcing in Oldroyd-B fluids with high Mach numbers.

We next discuss the physical mechanisms leading to large amplification. As evident
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Figure 4. Streamwise velocity fluctuations $v_1(z,y)$ containing the most variance in strongly elastic: (a) Couette flow; and (b) Poiseuille flow.

from both figure 2 and the expression for $\tilde{g}_0(k_z)$, the coupling term $C_{p2}$ plays an essential role in variance amplification; if this term was zero, the dynamics of weakly inertial/strongly elastic flows would be dominated by viscous dissipation (cf. figure 3(c) with $M = 0$). We note that $C_{p2}$ depends on the background shear $U'(y)$ and the spanwise/wall-normal variations in the flow fluctuations. A careful analysis of the governing equations (see Appendix C) demonstrates that this term emerges due to: (a) the wall-normal and spanwise velocity ($v_2, v_3$) gradients in the equation for $\phi_2 = \begin{bmatrix} \tau_{22} & \tau_{23} & \tau_{33} \end{bmatrix}^T$; (b) stretching of $\phi_2$ by the background shear in the equation for $\phi_4 = \begin{bmatrix} \tau_{12} & \tau_{13} \end{bmatrix}^T$; and (c) the $\phi_4$-gradients in the equation for $\omega_2$. All of these give rise to polymer stretching, leading to a transfer of energy from the base flow to fluctuations which results in large steady-state variance in weakly inertial/strongly elastic flows. The importance of this mechanism was recently recognized by Doering, Eckhardt & Schumacher (2006), where several examples of solutions to the two-dimensional equations for Couette flow of Oldroyd-B fluids that display transient growth were provided. Such energy transfer has also been observed experimentally in elastic turbulence of a swirling flow between two parallel disks (Groisman & Steinberg 2000, 2004). However, the present work demonstrates that this transfer can be initiated even in rectilinear flows.

Streamwise velocity flow structures that contain the most variance in strongly elastic flows with $k_z = 2.07$ (Couette) and $k_z = 2.24$ (Poiseuille) are shown in figure 4. These structures are purely harmonic in $z$ and their wall-normal shapes are determined by the principal eigenfunctions of operators $(k_z^2/4)\mathbf{T}_z^{-1}\mathbf{C}_{p2}\mathbf{T}_z^{-2}\mathbf{C}_{p2}^*\mathbf{T}_z^{-1}$ (Farrell & Ioannou 1993). The most amplified sets of fluctuations are given by high (hot colors) and low (cold colors) speed streaks. In Couette flow the streaks occupy the entire channel width, and in Poiseuille flow they are antisymmetric with respect to the channel’s centerline. These flow structures have striking resemblance to the initial conditions responsible for the largest transient growth in Newtonian fluids (Butler & Farrell 1992). Despite similarities, the fluctuations shown in figure 4 and in Butler & Farrell (1992) arise due to fundamentally different physical mechanisms: in high $Re$-flows of Newtonian fluids, vortex tilting is the main driving force for amplification; in high $\mu$/low $Re$-flows of viscoelastic fluids, it is the polymer stretching mechanism mentioned earlier. This suggests that polymer stretching in elasticity-dominated flows effectively introduces the lift-up of flow fluctuations in a similar manner as vortex tilting does in inertia-dominated flows (Landahl 1975).

5. Conclusions

In this work, we have analyzed nonmodal amplification of stochastic disturbances in streamwise-constant channel flows of Oldroyd-B fluids. For large elasticity numbers, the linearized governing equations can be decomposed into slow and fast subsystems, allowing
application of singular perturbation methods to obtain an analytical expression for the variance amplification associated with the velocity field. This expression agrees with that hypothesized by Hoda et al. (2009) on the basis of numerical data. For sufficiently large elasticity number, the variance amplification shows a peak at \(O(1)\) spanwise wavenumber, and the corresponding streamwise velocity fluctuations have a structure similar to that seen in high-Reynolds-number flows of Newtonian fluids. The mechanism of the energy amplification involves polymer stretching, which gives rise to an energy transfer from the base flow to fluctuations. This transfer can be interpreted as an effective lift-up of velocity fluctuations, similar to the role vortex tilting plays in inertia-dominated flows.

The present results further confirm our earlier observations (Hoda et al. 2008, 2009) that stochastic disturbances can be considerably amplified by elasticity even when inertial effects are weak. This amplification can serve as an initial stage of the development of high and low speed streaks, which upon reaching a finite amplitude may undergo secondary instability and thereby provide a bypass transition to elasticity-induced turbulence (Larson 2000; Groisman & Steinberg 2000, 2004; Berti et al. 2008). It is important to point out that although we have considered a particular class of disturbances in this work, our results raise the possibility that other types of disturbances might also be significantly amplified in elasticity-dominated flows. We note that in the present problem, energy amplification does not require the presence of curved streamlines in the base flow, which can give rise to linear instabilities in other geometries when the effects of elasticity dominate those of inertia (Larson 1992). (In the present problem, the base flow is linearly stable (Larson 1992).) However, the finite-amplitude flow structures created by the energy amplification explored here may well contain curved streamlines and be subject to further instabilities that lead to a disordered flow. Elasticity-induced turbulence may find use in promoting mixing in microfluidic devices, where inertial effects are weak due to the small geometries (Groisman & Steinberg 2001, 2004). In polymer processing applications, however, elasticity-induced turbulence is generally undesired (Larson 1992, 2000), and our work may aid the development of control strategies to maintain ordered flows.

Finally, we point out that the slow-fast decomposition of the linearized dynamics we have uncovered here does not follow \textit{a priori} from the governing equations in their original form. Identification of this decomposition was a necessary step in the application of the singular perturbation methods that were used to develop an analytical expression for the variance amplification. The approach taken in this work may be helpful in examining the asymptotic structure of other flows at high elasticity number, especially if such flows are subject to disturbances and have linearized governing equations which are nonnormal in nature.

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\textbf{Appendix A. Governing equations}

The non-dimensional momentum conservation, mass conservation, and constitutive equations for an Oldroyd-B fluid are given by (Bird et al. 1987; Larson 1999)

\[
\begin{align*}
\textbf{V}_t & = \frac{1}{\text{Re}} (\beta \nabla^2 \textbf{V} + (1 - \beta) \nabla \cdot \textbf{T} - \nabla P) - \nabla \cdot \textbf{V} + \vec{F}, \\
0 & = \nabla \cdot \textbf{V}, \\
\textbf{T}_t & = \frac{1}{\text{We}} (\nabla \textbf{V} + (\nabla \textbf{V})^T - \textbf{T}) - \nabla \cdot \textbf{T} + \nabla \textbf{T} + (\textbf{T} \cdot \nabla \textbf{V})^T.
\end{align*}
\]
where \( \mathbf{V} \) is the velocity vector, \( P \) is the pressure, \( \mathbf{T} \) is the polymeric contribution to the stress tensor, and \( \mathbf{F} \) is the spatio-temporal body force per unit mass. These equations have been brought to a dimensionless form by scaling length with the channel half height \( L \), velocity with the largest base velocity \( U_o \), time with \( L/U_o \), polymer stresses with \( \eta_p U_o / L \), pressure with \( (\eta_\ast + \eta_p) U_o / L \), and body force with \( U_o^2 / L \).

In flows with high elasticity numbers, \( \epsilon = 1/\mu \ll 1 \), it is more convenient to rescale time as \( t = \tilde{t} / W e \), which leads to the following set of equations:

\[
\begin{align*}
\epsilon \mathbf{V}_t &= \beta \mathbf{V}^2 + (1 - \beta) \mathbf{V} \cdot \mathbf{T} - \mathbf{V} P - Re \nabla \mathbf{V} \mathbf{V} + \sqrt{\epsilon Re} \mathbf{F}, \\
0 &= \mathbf{V} \cdot \mathbf{V}, \\
\mathbf{T}_t &= \nabla \mathbf{V} + (\nabla \mathbf{V})^T - \mathbf{T} + \frac{Re}{\epsilon} (\mathbf{T} \cdot \nabla \mathbf{V} + (\mathbf{T} \cdot \nabla \mathbf{V})^T - \nabla \mathbf{V} \mathbf{T}).
\end{align*}
\] (A2)

Since we are interested in considering problems with stochastic spatio-temporal excitation, the body-forces in equations (A2) and (A1) are related to each other by \( \mathbf{F}(\mathbf{r}, t) = \sqrt{W e} \tilde{\mathbf{F}}(\mathbf{r}, t W e) \), where \( \mathbf{r} \) denotes the vector of spatial coordinates; this scaling is introduced to guarantee the same auto-correlation operators of \( \mathbf{F}(\mathbf{r}, t) \) and \( \tilde{\mathbf{F}}(\mathbf{r}, \tilde{t}) \) (Kokotović et al. 1999).

Linearized dynamics are given by (2.1) and they are obtained by decomposing each field in (A 2) into the sum of the base flow and fluctuations (i.e., \( \mathbf{V} = \mathbf{v} + \mathbf{v} \), \( \mathbf{T} = \mathbf{\tau} + \mathbf{\tau} \), \( P = \tilde{p} + p \), \( \mathbf{F} = 0 + \mathbf{d} \)), and keeping terms only to first order in fluctuations.

### Appendix B. The streamwise-constant evolution model

The evolution model of the linearized system is obtained by a standard conversion to the wall-normal velocity/vorticity \((v_2, \omega_2)\) formulation. The procedure described in Hoda et al. (2008) in combination with the Fourier transform in \( z \) converts system (2.1) with streamwise-constant fluctuations \( (\partial_z \cdot \cdot \cdot) = 0 \) to:

\[
\begin{align*}
\epsilon \dot{\phi}_1 &= \beta S_{11} \phi_1 + (1 - \beta) S_{12} \phi_2 + \sqrt{\epsilon Re} (F_{21} d_2 + F_{31} d_3), \quad \text{(B1a)} \\
\phi_2 &= -\phi_2 + S_{21} \phi_1, \quad \text{(B1b)} \\
\epsilon \dot{\phi}_3 &= \beta S_{33} \phi_3 + Re S_{31} \phi_1 + (1 - \beta) S_{34} \phi_4 + \sqrt{\epsilon Re} F_{11} d_1, \quad \text{(B1c)} \\
\dot{\phi}_4 &= -\phi_4 + (Re/\epsilon) (S_{41} \phi_1 + S_{42} \phi_2) + S_{43} \phi_3, \quad \text{(B1d)} \\
\dot{\phi}_5 &= -\phi_5 - (Re/\epsilon)^2 S_{51} \phi_1 + (Re/\epsilon) (S_{53} \phi_3 + S_{54} \phi_4), \quad \text{(B1e)} \\
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 & G_1 & 0 \\ G_2 & 0 & 0 \\ G_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \quad \text{(B1f)}
\end{align*}
\]

where \( \phi_1 = v_2, \phi_2 = [\tau_{22} \tau_{23} \tau_{33}]^T, \phi_3 = \omega_2, \phi_4 = [\tau_{12} \tau_{13}]^T, \phi_5 = \tau_{11} \). The \( S \)-operators are given by:

\[
\begin{align*}
S_{11} &= \Delta^{-1} \Delta^2, \quad S_{33} = \Delta, \quad S_{31} = -i k_z U'(y), \\
S_{12} &= \Delta^{-1} \left[ -k_z^2 \partial_y - i k_z (\partial_{yy} + k_z^2) \right], \quad S_{34} = \left[ i k_z \partial_y - k_z^2 \right], \\
S_{21} &= \left[ 2 \partial_y (i/k_z) (\partial_{yy} + k_z^2) - 2 \partial_y \right]^T, \quad S_{43} = -(1/k_z^2) S_{34}, \\
S_{41} &= \left[ U'(y) \partial_y - U''(y) \right], \quad S_{42} = \left[ U'(y) 0 0 \right], \\
S_{51} &= 4U'(y) U''(y), \quad S_{53} = -(2i/k_z) U'(y) \partial_y, \quad S_{54} = \left[ 2U'(y) 0 \right].
\end{align*}
\]
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The expressions for operators $\Delta, \Delta^2, F_j,$ and $G_3,$ are given in § 2.1. Note that (B 1) represents a system of PDEs in the wall-normal direction and in time parameterized by $k_z, Re, \beta,$ and $\epsilon.$

Appendix C. Frequency response operators

The frequency response operators $H_{ij},$ relating the forcing and velocity components, $v_i = H_{ij} d_j,$ can be obtained by applying the temporal Fourier transform to (B 1). Equation (B 1b) can be used to express $\phi_2 = [\tau_{22} \tau_{23} \tau_{33}]^T$ in terms of the wall-normal velocity $\phi_1 = v_2$

$$\phi_2 = \frac{S_{11} - 1}{1 + \beta} \phi_1,$$

where $\omega$ denotes the temporal frequency. Substitution of (C 1) into the temporal Fourier transform of (B 1a) yields

$$\phi_1 = \sqrt{\epsilon Re} \left( \epsilon i \omega I - \beta S_{11} - \frac{1 - \beta}{1 + \omega} S_{12} S_{21} \right)^{-1} \left( F_2 d_2 + F_3 d_3 \right).$$

Based on this and equation (B 1f), it follows that, for streamwise constant fluctuations, streamwise forcing does not influence the wall-normal and spanwise velocities, i.e.

$$H_{i1}(k_z, \omega; Re, \beta, \epsilon) = 0, \quad i = 2, 3.$$ 

Moreover, using (B 1f) and the fact that $S_{12} S_{21} = S_{11} = \Delta^{-1} \Delta^2 =: T_2,$ the operators $H_{ij}(k_z, \omega; Re, \beta, \epsilon), \{i, j = 2, 3\},$ can be written as

$$H_{ij}(k_z, \omega; Re, \beta, \epsilon) = \sqrt{Re} H_{ij}(k_z, \omega; \beta, \epsilon), \quad i, j = 2, 3,$$

where the $Re$-independent operators $H_{ij}$ are given by

$$H_{ij}(k_z, \omega; \beta, \epsilon) = \sqrt{\epsilon} (1 + i \omega) G_i K_j F_j,$$

$$K_2 = (\epsilon i \omega)^2 I - (\beta T_2 - \epsilon I) i \omega - T_2^{-1}.$$

The following relation between the wall-normal velocity/vorticity $(\phi_1, \phi_3)$ and polymer stresses $\phi_4 = [\tau_{12} \tau_{13}]^T$ can be established by substituting (C 1) into the temporal Fourier transform of (B 1d)

$$\phi_4 = \frac{(Re/\epsilon)}{1 + i \omega} \left( S_{41} + S_{42} S_{21} \right) \phi_1 + \frac{1}{1 + i \omega} S_{43} \phi_3.$$

Substitution of this equation in the temporal Fourier transform of (B 1c) yields

$$\phi_3 = Re \left( \epsilon i \omega I - \beta S_{33} - \frac{1 - \beta}{1 + \omega} S_{34} S_{43} \right)^{-1} \left( S_{31} + \frac{1 - \beta}{\epsilon (1 + i \omega)} \left( S_{41} + S_{42} S_{21} \right) \right) \phi_1$$

$$+ \sqrt{\epsilon Re} \left( \epsilon i \omega I - \beta S_{33} - \frac{1 - \beta}{1 + \omega} S_{34} S_{43} \right)^{-1} F_1 d_1.$$

Now, since $v_1 = G_1 \phi_3,$ by substituting (C 2) into (C 3) and using the fact that in Couette and Poiseuille flows $S_{34} S_{43} = S_{33} = \Delta =: T_1, S_{34} S_{41} = 0, S_{34} S_{42} S_{21} = i k_z (U''(y) \Delta + 2 U''(y) \partial_y),$ it follows that operators $H_{ij}(k_z, \omega; Re, \beta, \epsilon), \{j = 1, 2, 3\},$ are given by

$$H_{i1}(k_z, \omega; Re, \beta, \epsilon) = \sqrt{Re} H_{i1}(k_z, \omega; \beta, \epsilon),$$

$$H_{1j}(k_z, \omega; Re, \beta, \epsilon) = \sqrt{Re}^3 H_{1j}(k_z, \omega; \beta, \epsilon), \quad j = 2, 3.$$
Here, the $Re$-independent operators $\hat{H}_{ij}$ are determined by

$$
\hat{H}_{11}(k_z, \omega; \beta, \epsilon) = \sqrt{\epsilon}(1 + i \omega) G_1 K_1 F_1,
$$

$$
\hat{H}_{1j}(k_z, \omega; \beta, \epsilon) = \frac{1}{\sqrt{\epsilon}} G_1 K_1 (\epsilon(1 + i \omega)^2 C_{p1} + C_{p2}) K_2 F_j, \quad j = 2, 3,
$$

with $K_1 = (\epsilon(i \omega)^2 I - (\beta T_1 - i I) \omega - T_1)^{-1}$, $C_{p1} = S_{31} = -ik_z U'(y)$, and $C_{p2}$ is $ik_z(1 - \beta)\bar{C}_{p2}$, $\bar{C}_{p2} = U'(y) \Delta + 2U''(y) \partial_y$. Clearly, $ik_z\bar{C}_{p2} = S_{34} S_{42}$, which indicates that operator $C_{p2}$ arises from: (a) the wall-normal and spanwise velocity $(v_2, v_3)$ gradients, $S_{21}$, in the equation for $\phi_2$; (b) stretching of $\phi_2$ by the background shear, $S_{42}$, in the equation for $\phi_4$; and (c) the $\phi_4$-gradients, $S_{34}$, in the equation for $\phi_3$.

**Appendix D. Scaling of functions $f_{ij}$ and $g_{ij}$ with $\mu$**

In this appendix, we briefly describe how equation (3.7) can be combined with the perturbation analysis of §3 to uncover the scaling of variance amplification with $\mu$ in elasticity-dominated flows. The key observation here is that each operator in (3.7) exhibits a power series expansion, e.g. $X(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n X_n$. Thus, equation (3.7) can be rewritten as

$$
E_{ij} = r^2(\epsilon) \text{trace}(\Psi(\epsilon)) + (r^2(\epsilon)/\epsilon) \text{trace}(\Phi(\epsilon)), \quad (D1)
$$

where

$$
\Psi(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Psi_n = X(\epsilon) C_s^*(\epsilon) C_s(\epsilon) + Y(\epsilon) C_s^*(\epsilon) C_f(\epsilon) + C_f^*(\epsilon) C_s(\epsilon) Y^*(\epsilon),
$$

$$
\Phi(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Phi_n = Z(\epsilon) C_f^*(\epsilon) C_f(\epsilon).
$$

In what follows, $f_{ij} = E_{ij}$ for $\{i = 1; i, j = 2, 3\}$ and $g_{ij} = E_{ij}$ for $\{i = 1; j = 2, 3\}$.

**D.1. Scaling of functions $f_{ij}$ with $\mu$**

We next examine the variance amplification of operators $\hat{H}_{ij}$ with $\{i = j = 1; i, j = 2, 3\}$. For each of these operators $r(\epsilon) = \sqrt{\epsilon}$ and, thus, equation (D1) simplifies to

$$
f_{ij} = \epsilon \sum_{n=0}^{\infty} \epsilon^n \text{trace}(\Psi_n) + \sum_{n=0}^{\infty} \epsilon^n \text{trace}(\Phi_n).
$$

The expression for function $f = \sum_{i,j} f_{ij}$ given in §3.1 is a direct consequence of this equation. In particular, we have

$$
f(k_z; \beta, \epsilon) = \sum_{i,j} f_{ij}(k_z; \beta, \epsilon) = \sum_{i,j} \hat{f}_{ij,0}(k_z; \beta) + O(\epsilon),
$$

where $\epsilon$-independent functions $\hat{f}_{ij,0}(k_z; \beta)$ are determined by

$$
\hat{f}_{ij,0}(k_z; \beta) = \text{trace}(\Phi_0) = \text{trace}(Z_0 C_{f,0}^* C_{f,0}),
$$

$$
A_{f,0} Z_0 + Z_0 A_{f,0}^* = -B_f B_f^*.
$$

Note that the Lyapunov equation for $Z_0$ follows from the perturbation analysis of §3 and that $C_{f,0}$ is indexed by $i$, whereas $A_{f,0}$ and $B_f$ (and consequently $Z_0$) are indexed by $j$

$$
A_{f,0} = \beta T_k, \quad \{k = 1 \text{ for } j = 1; k = 2 \text{ for } j = 2, 3\}, \quad B_f = F_j, \quad C_{f,0} = G_i.
$$
A direct comparison of the above expressions with their Newtonian fluid equivalents (see Bamieh & Dahleh 2001; Jovanović & Bamieh 2005) yields

\[ \tilde{f}_{ij,0}(k_z; \beta) = \tilde{f}_{ij,0}(k_z)/\beta. \]

Here, \( \tilde{f}_{ij,0}(k_z) \) denotes variance amplification from \( d_j \) to \( v_i \), \( \{ i = j = 1; i, j = 2, 3 \} \), in Newtonian fluids with \( Re = 1 \). Thus, in flows with high elasticity number, the term responsible for the \( Re \)-scaling in (E) is inversely proportional to \( \beta \) and approximately \( \mu \)-independent. Finally, as shown in Bamieh & Dahleh (2001); Jovanović & Bamieh (2005),

\[ \tilde{f}_0(k_z) = \sum_{i,j} \tilde{f}_{ij,0}(k_z) = -0.5(\text{trace}(T_1^{-1}) + \text{trace}(T_2^{-1})). \]

The \( k_z \)-dependence of this base-flow-independent function is shown in figure 3(a).

D.2. Scaling of functions \( g_{ij} \) with \( \mu \)

Analysis of \( \tilde{H}_{12} \) and \( \tilde{H}_{13} \) can be used to determine the \( \mu \)-dependence of terms responsible for the \( Re \)-scaling of the steady-state velocity variance. As observed in § 3.2, for both these operators \( C_s = C_1 \) is \( \epsilon \)-independent, \( C_f(\epsilon) = \epsilon C_1 Q(\epsilon) \), and \( r(\epsilon) = 1/\sqrt{\epsilon} \). Thus, operators \( \Psi(\epsilon) \) and \( \Phi(\epsilon) \) in (D 1) can be written as

\[
\Psi(\epsilon) = X(\epsilon)C_1^*C_1 + \epsilon (Y(\epsilon)C_1^*C_1 Q(\epsilon) + Q^*(\epsilon)C_1^*C_1 Y^*(\epsilon)),
\]

\[
\Phi(\epsilon) = \epsilon^2 Z(\epsilon)Q^*(\epsilon)C_1^*C_1 Q(\epsilon).
\]

These two expressions in conjunction with \( r(\epsilon) = 1/\sqrt{\epsilon} \) and equation (D 1) can be used to obtain

\[ g_{ij} = (1/\epsilon) \text{trace}(X(\epsilon)C_1^*C_1) + \text{trace}(Z(\epsilon)Q^*(\epsilon)C_1^*C_1 Q(\epsilon)) + \text{trace}(Y(\epsilon)C_1^*C_1 Q(\epsilon) + Q^*(\epsilon)C_1^*C_1 Y^*(\epsilon)), \]

where \( X(\epsilon), Y(\epsilon), \) and \( Z(\epsilon) \) are indexed by \( j \). As noted before, each of the operators in the equation for \( g_{ij} \) can be decomposed into a power series expansion, which in turn yields,

\[ g(k_z; \beta, \epsilon) = \sum_{j=2}^{3} g_{ij}(k_z; \beta, \epsilon) = \frac{1}{\epsilon} \sum_{n=0}^{\infty} \epsilon^n \hat{g}_n(k_z; \beta), \quad \epsilon \ll 1, \]

where each \( \hat{g}_n(k_z; \beta) \) is \( \epsilon \)-independent. In particular, at high \( \mu \) we have

\[ g(k_z; \beta, \epsilon) = (1/\epsilon)\hat{g}_0(k_z; \beta) + O(1) = (1/\epsilon) \sum_{j=2}^{3} \text{trace}(X_{j,0} C_1^* C_1) + O(1). \]

A careful analysis of the Lyapunov equation for \( X_{j,0} \),

\[
A_{s,0} X_{j,0} + X_{j,0} A_{s,0}^* = -B_{j,0} B_{j,0}^*, \quad j = 2, 3,
\]

in combination with the structure of operator \( C_1 \) can be used to finally obtain

\[ \hat{g}_0(k_z; \beta) = \hat{g}_0(k_z)(1 - \beta)^2/\beta, \quad \hat{g}_0(k_z) = (k_z^2/4) \text{trace}(T_1^{-1} \tilde{C}_{p2} T_2^{-2} \tilde{C}_{p2}^* T_1^{-1}). \]

An in-depth study of function \( \hat{g}_0(k_z) \) and its importance in early stages of transition to elasticity-induced turbulence is provided in § 4.
REFERENCES


