Nonlinear Systems

Lecture 27  
05/07/13

Last time

- Input-output linearization
- Zero-dynamics

\[ y = \text{position of cart} \]

Zero dynamics: dynamics of inverted pendulum.

Example of a system that doesn't have a globally defined relative degree:

\[ \begin{align*}
\dot{x}_1 &= x_2 + x_3^3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u \\
y &= x_1 \\
\end{align*} \]

\[ \begin{align*}
\dot{y} &= \dot{x}_1 = x_2 + x_3^3 \\
\dot{y} &= \dot{x}_2 + 3x_2^2 \dot{x}_3 = x_3 + 3x_3^2 u \\
\end{align*} \]
\[ \dot{y} = x_3 + 3x_3^2 u \]

vanishes when \( x_3 = 0 \)

this system doesn't have well defined relative degree

this was an example of a system that doesn't have a well-defined relative degree (\( x_3 = 0 \))

when \( x_3 = 0 \) we would not be able to determine the influence of \( u \), input, on \( \dot{y} \) or the derivative of \( y \).

recall: \[ \begin{array}{c}
\n \end{array} \]

\[ y^{(m)} = L^f h(m) + \frac{Lg L_{m}^{-1} h(m) \cdot u}{\neq 0} \]

\[ u = \frac{1}{Lg L_{m}^{-1} h(m)} \left\{ -L^f h(m) + \dot{y} \right\} \]

useful framework for output tracking that shows limitations in ideal conditions. However, issue of robustness can be problematic when non-linearities are not well characterized.
"Normal Form"

Translation: set of coordinates that displays zero dynamics.

\[ \begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*} \]

If the above system has a well-defined relative degree \( r \leq n \) then there exist new variables

\[ \begin{align*}
z &\in \mathbb{R}^{n-r} \\
\xi &\in \mathbb{R}^{r}
\end{align*} \]

such that \( T(x) = [z] ; T(0) = 0 \) is a diffeomorphism (\( T^{-1} \) exists ; \( T, T^{-1} \) are cts, diffble) and the dynamics in new coordinates have the form:

\[ \begin{align*}
\dot{z} &= f_0(z, \xi) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
&\vdots \\
\dot{\xi}_r &= b(z, \xi) + a(z, \xi)u
\end{align*} \]
In particular:

\[ \mathbf{x} := \begin{bmatrix} h(x) \\ \dot{h}(x) \\ \vdots \\ \dot{h}^{(n-1)}(x) \end{bmatrix}, \quad b(z, x) = L_f h(x) \\
\quad a(z, x) = L_f \dot{h}^{(n-1)}(x) \]

and \( z = f_0(z, 0) \) is the zero dynamics.

\( z \) should be linearly indep. of \( \mathbf{x} \) and achieve \( \dot{z} = f(z, x) \). 
\( \mathbf{x} \) variables represent the output

I/O linearizing controller:

\[ u = \frac{1}{a(z, x)} (-b(z, x) + v) \quad \ldots \quad (1) \]

\[ v = -K_1 x_1 - \cdots - K_r x_r \quad \ldots \quad (2) \]

gives

\[ \dot{x} = A_f x \quad \text{where} \quad A_f = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -K_1 & -K_2 & -K_3 & \cdots & -K_r \end{bmatrix} \]

is in the companion (controller-canonical) form & is Hurwitz!
If our objective is local asymptotic stability, then we can use linearization of (*) with (1), (2):

$$ \begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_0}{\partial z} |_{0} & \frac{\partial f_0}{\partial \xi} |_{0} \\ 0 & A_\xi \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} $$

Therefore, the stability of (local asymptotic stability) this matrix is equivalent to $\frac{\partial f_0}{\partial z} |_{0}$ being Hurwitz (stability of zero dynamics).

If you have input to state stability you will have global asymptotic stability as well.

$\xi$ is not important at this point when studying local stability properties.
Ex
\[ \dot{x}_1 = x_2 \]  \hspace{1cm} (1)
\[ \dot{x}_2 = ax_3 + u \]  \hspace{1cm} (2)
\[ \dot{x}_3 = bx_3 - u \]  \hspace{1cm} (3)
\[ y = x_1 \]
\[ \dot{y} = x_1 = x_2 \]
\[ \ddot{y} = x_2 = ax_3 + u \]

Relative degree is 2. (globally well defined relative degree since nothing comes in front of \( u \))

\[ \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \dot{s}_1 = s_2 \]
\[ \dot{s}_2 = ax_3 + u \]

Now to determine zero dynamics → \( x_3 \) by itself is not enough b/c input should not enter into zero dynamics.
Note:

a) We have identified $f$-part of the system, but still need to find $\dot{z} = f_c(z, s)$.

b) $x_3 \neq z$ (because $x$ enters into the equations for $x_3$)

\[
\begin{align*}
  y = 0 & \implies \dot{y} = 0 \implies \dot{s}_1 = 0 \\
  \dot{y} = 0 & \implies \dot{s}_2 = 0 \\
  \downarrow & \\
  u = -\alpha x_3 & \text{(for } y = 0) \\
  \implies \dot{x}_3 = \beta x_3 - u \\
  \implies \dot{x}_3 = (\alpha + \beta) x_3
\end{align*}
\]

Stability determined by sign of $(\alpha + \beta)$:

\[
\begin{align*}
  \alpha + \beta & > 0 \implies \text{unstable} \\
  \alpha + \beta & < 0 \implies \text{locally asymptotically stable}
\end{align*}
\]

Construction of $z$:

(2) + (3) \implies \frac{d(z_2 + x_3)}{dt} = (\alpha + \beta) x_3

\[
\begin{align*}
  z &= x_2 + x_3 \\
  x_3 &= z - x_2 \\
  & \implies \dot{z} = (\alpha + \beta) z - (\alpha + \beta) x_2
\end{align*}
\]