Nonlinear Systems

Lecture 25

04/30/13

Input to state stability

Linear Systems

\[ \dot{x} = Ax + Bu \]

stability of \( \dot{x} = Ax \) guarantees boundedness of the state for bounded inputs

\[ x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \]

Vector norm

\[ \| x(t) \| \leq \| e^{At} \| \| x_0 \| + \int_0^t \| e^{A(t-\tau)} \| \| Bu(\tau) \| d\tau \]

induced norm
(largest singular value of matrix exp.)

\[ \leq Ke^{-\alpha t} \| x_0 \| + \| B \| \sup_{0 \leq \tau \leq t} \| u(\tau) \| \int_0^t Ke^{-\alpha \tau} d\tau \]
\[ \|x(t)\| \leq KE^{-t}\|x_0\| + K_\alpha \|B\| \sup_{0 \leq \tau \leq t} \|u(\tau)\| \]

\( \quad \text{effect of initial cond.} \quad \text{effect of inputs} \)

Derived under the assumption that \( \dot{x} = Ax \) is stable (eigenvalues of \( A \) in the LHP)

\[ x \quad \downarrow \quad x \quad \downarrow \quad x \]

Unfortunately, for nonlinear systems this property doesn't hold.

**Ex.** (Counter example)

\[ \dot{x} = -x + xu \]

Note: \( \dot{x} = -x \) is stable but any \( u \) with \( |u(t)| > 1 \) is going to generate unbounded response.

Eg. \( u(t) = 2 \)

\[ \dot{x} = -x + x \cdot 2 \Rightarrow \dot{x} = x \Rightarrow x(t) = e^t x_0 \]
Def. A system $\dot{x} = f(x,u)$ is input-to-state stable (ISS) if:

$$\|x(t)\| \leq \beta(\|x_0\|,t) + \gamma(\sup_{0 \leq \tau \leq t} \|u(\tau)\|)$$

For linear systems

$$\beta(r,t) = K e^{-\alpha t}$$

$$\gamma(s) = \frac{K}{\alpha} \|B\| \|s\|$$

Implications of ISS:

1) $\dot{x} = f(x,u)$ is ISS $\Rightarrow$ $\dot{x} = f(x,0)$ is globally asymptotically stable.

2) If $u(t) \xrightarrow{t \to \infty} 0$ $\Rightarrow$ $x(t) \xrightarrow{t \to \infty} 0$

A dissipation-like inequality for ISS:

If there are class $K_\infty$ functions $\alpha_i(\cdot)$; $i=1,2,3,4$ and a cts differentiable function $V(x)$ st.
\[ \alpha_1 (\|x\|) \leq V(x) \leq \alpha_2 (\|x\|) \]

\[ \dot{V} = \frac{\partial V}{\partial x} f(xu) \leq -\alpha_3 (\|x\|) + \alpha_4 (\|u\|) \]

**Proof** (Khalil)

\[ \exists - \dot{x} = -x^p + x^q u \]

\[ p \text{ is an odd integer} \]

\[ \text{ISS if } p > q \]

\[ V(x) = \frac{1}{2} x^2 \Rightarrow \dot{V} = x \cdot \dot{x} = -x^{p+1} + x^{q+1} u \]

* Young's inequality

\[ a \cdot b \leq \frac{a^r}{r} |x|^r + \frac{1}{s a^s} |b|^s \]

\[ r > 1 \quad \& \quad (r-1)(s-1) = 1 \quad \& \quad \alpha > 0 \]

\[ \Rightarrow \dot{V} = -x^{p+1} + x^{q+1} u \]

\[ x^{q+1} u \leq \frac{\alpha^r}{r} |x|^{(q+1)r} + \frac{1}{s \alpha^s} |u|^s \]

\[ 4 \]
Choose: \( r = \frac{p+1}{q+1} > 1 \); \( s = 1 + \frac{1}{r-1} \)

and \( \alpha \) st. \( \frac{\alpha r}{r} = \frac{1}{2} \)

\[
\dot{V} \leq -\frac{1}{2} |x|^p + \frac{1}{5 \alpha s^s} |u|^s
\]

\( \alpha_y(1x1) \quad \alpha_z(1u11) \)

Note! \( p \) has to be strictly larger than \( q \)
(otherwise no ISS, i.e., \( x = -x + xu \))

Feedback Linearization

Important notions: input-output linearization
relative degree
gzero dynamics

\[
\dot{x} = f(x) + g(x)u \\
y = h(x)
\]

\( \{u(t) \in \mathbb{R} \} \text{ scalars} \quad \{y(t) \in \mathbb{R} \} \quad (\text{SISO nonlinear system}) \)
relativ degree:

Number of times that we need to differentiate the output to "see" the input (for input to appear in the output eqn)

\[ y = h(x) \implies \dot{y} = \frac{d}{dx} \hat{x} = \frac{d}{dx} (f(x) + g(x)u) \]

\[ = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)u \]

\[ \text{L}_f h(x) \quad \text{L}_g h(x) \]

Lec derivative of func h in direction of f

If \( \text{L}_g h(x) \neq 0 \) in an open set containing the equilibrium then relative degree (r.d.) = 1

If not keep differentiating

\[ \ddot{y} = \frac{d}{dx} \hat{x} = \frac{d}{dx} (f(x) + g(x)u) \]

\[ \text{L}_f^2 h(x) \quad \text{L}_g^2 h(x) \]

\[ \text{same as before} \]
Def. System \( \dot{x} = f(x) + g(x)u \) has relative degree \( r \) if in

\[
y = h(x)
\]

a neighborhood of the equilibrium:

\[
\begin{align*}
Lg L^i f h(x) &= 0, & i = 1, 2, \ldots, r-1 \\
Lg L^r f h(x) &\neq 0
\end{align*}
\]

Ex

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 + u
\end{align*}
\]

\[
y = x_1
\]

\[
\dot{y} = \dot{x}_1 = x_2 \quad \text{no input} \quad \rightarrow \text{keep differentiating}
\]

\[
\text{means r.d. at least bigger than one}
\]

\[
\dot{\dot{y}} = \dot{x}_2 = -x_1^3 + u \quad \rightarrow \text{r.d.} = 2
\]

\[
\dot{\dot{y}} = -\dot{y}^3 + u
\]

now easy to deal

\[
\text{with}
\]

\[
\text{block diagram}
\]
choose \( u = -k_1 y - k_2 y + y^3 \) → bad from design point of view but give linear dynamic response

Linearization by means of Taylor.

Ex2
\[
\begin{align*}
y &= x_2 \\
y &= \dot{x}_2 = -x_1^3 + u
\end{align*}
\]
r.d. 1