Nonlinear System

Lecture 23

\[ \int_0^T y^T(t) u(t) \, dt \geq 0 \quad \text{(standard notion of passivity)} \]

has to hold for all times \( T \) and all input trajectories accounts for LQG's

\[ \int_0^T y^T(t) u(t) \, dt \geq \begin{cases} -\beta \\ \sum \langle u_t, u_t \rangle - \beta \\ \sum \langle y_t, y_t \rangle - \beta \end{cases} \quad \text{passive (P)} \]

input strictly passive (ISP) \quad \text{output strictly passive (OSP)}

→ State space characterization

\[ \dot{V}(x) = \begin{bmatrix} y^T u \\ -\delta u + y^T u \\ -e_y y + y^T u \end{bmatrix} \quad \text{(ISP)} \]

\[ V = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial u} g(x) \, u \quad h^T(x) \, u \quad \text{(*)} \]

\[ \text{(*)} \iff \frac{\partial V}{\partial x} f(x) \leq 0 \quad (1) \rightarrow \text{stability in the sense of Lyap.} \]

additional condition

\[ \frac{\partial V}{\partial u} g(x) = h^T(x) \quad (2) \]
For linear systems
\[ V(x) = \frac{1}{2} x^T P x \]
\[ A^T P + P A < 0 \quad (1) \]
\[ P B = C^T \quad (2) \]

Implications of positive realness:

1) If \( H(s) \) is PR \( \Rightarrow \) stable

2) \( |\angle H(j\omega)| < 90^\circ \) (Bode plot)

Nyquist plot lies in the right half plane

3) Relative degree of \( H \) is either 0 or 1

\[ H(s) = \frac{P(s)}{Q(s)} \quad \text{order of } P - \text{order of } Q = (\text{# of poles}) - (\text{# of zeros}) = \text{relative degree} \]
KYP Lemma (Kalman, Yakubovich, Popov)  
(Positive Real Lemma) 

Let \( H(s) = C (sI - A)^{-1} B \) with 

\[
\Re \lambda(A) < 0 \quad \text{and} \quad (A,B) \text{ controllable}
\]

then \( H(s) \) is PR iff \( EP = PT > 0 \) st. \( AT + PA < 0 \) and \( PB = CT \)

\( H(s) \) is SPR iff \( EP = PT > 0 \) st. \( AT + PA < 0 \), 
\[ PB = CT \]

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Passivity thm.

\[
\begin{array}{c}
H: [u_1] \\
\downarrow \square \\
H_1 \\
\downarrow \square \\
H_2 \\
\downarrow \square \\
y_1 \\
\end{array}
\]

\[
\begin{array}{c}
H: [u_1] \\
\downarrow \square \\
H_1 \\
\downarrow \square \\
H_2 \\
\downarrow \square \\
y_2 \\
\end{array}
\]

\( a) \quad H_1: \text{ passive with storage functions } V_1 \& V_2 
\]

\[
\begin{align*}
H_1: \quad & \dot{V}_1 \leq e_1^T \dot{y}_1 = (u_1 - y_2) \dot{y}_1 = u_1 \dot{y}_1 - y_2 \dot{y}_1, \quad (1) \\
H_2: \quad & \dot{V}_2 \leq e_2^T \dot{y}_2 = (u_2 + y_1) \dot{y}_2 = u_2 \dot{y}_2 + y_1 \dot{y}_2, \quad (2)
\end{align*}
\]
(1) + (2) with $V := V_1 + V_2$

\[ \dot{V} = \dot{V}_1 + \dot{V}_2 \leq u_1^T y_1 + u_2^T y_2 = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix} \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix} = u^T y \]

This characterization is very general. \textit{H} (transfer functions) can be anything, even infinite dimensional.

b) $H_1$, same as in (a) but now let $H_2$ be memoryless nonlinearity with:

\[ H_2: \begin{bmatrix} y_2^T e_2 \end{bmatrix} > 0 \]

\[ \dot{y}_2^T (y_1 + u_2) \geq 0 \]

\[ -y_2^T y_1 \leq y_2^T u_2 \quad (3) \]

(3) $\rightarrow$ (1)

\[ \dot{V}_1 \leq u_1^T y_1 + u_2^T y_2 = u^T y \]

$\Rightarrow$ feedback interconnection with storage function $V_1$. 

\[ \text{4} \]
Linear Systems:

\[ H_1 \]

\[ H_2 \]

Recall: small-gain

\[ |H_1(j\omega)||H_2(j\omega)| < 1 \quad \text{for every } \omega \]

Passivity? \quad \Rightarrow \quad \text{info about phase characterizations}

\[ H_1: \text{positive real} \quad \angle H_1(j\omega) < 90^\circ \]

\[ \angle \left| H_1(j\omega)H_2(j\omega) \right| < 180^\circ \Rightarrow \]

Nyquist plot doesn't cross the real line, i.e. doesn't encircle -1.
Side note: inverse optimality (Kalman '62-63)

LQR:

\[ K = R^* B^T P \]

\[ \text{let } R = I \text{ (for simplicity)} \]

\[ K_{\text{opt}} = B^T P I \implies PB = K^T_{\text{opt}} \]

ARE:

\[ A^T P + PA + Q - PBR^T B^T P = 0 \]

Kalman showed that a given feedback gain \( K \) is inversely optimal (\( Q \& R \) can be recovered) if passivity holds when output matrix \( C \) is chosen to be \( K \).

\[ (A - BK)^T P + P(A - BK) = -Q - K^T R K \]

(RE for open loop equation is equivalent to the Lyapunov equation for the closed loop system.)
A generalization to (potentially) large-scale interconnections:

\[ \dot{H}_i: \quad \dot{x}_i = f(x_i) + g(x_i)u_i \]
\[ y_i = h(x_i) \]

\[ H_i: \quad \text{SISO} \]

Each \( H_i \): output strictly passive

\[ \dot{V}_i \leq -\varepsilon_i y_i^2 + y_i u_i \]

\[ u := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad y := \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad u = Ky \]

let

\[ V = \sum_{i=1}^{n} d_i V_i \quad ; \quad d_i > 0 \]
\[ \dot{V} \leq \sum_{i=1}^{n} d_i (-\varepsilon_i y_i^2 + y_i u_i) \]
We want to derive sufficient conditions for $K$ for the stability of the interconnection

$$D_d := \text{diag}\{d_i\} = \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix}$$

$$D_q := \text{diag}\{\varepsilon_i\} = \begin{bmatrix} \varepsilon_1 & \cdots & \varepsilon_n \end{bmatrix}$$

$$V < y^T (D_d \cdot D_\varepsilon) y + y^T D_d K y = y^T (-D_d D_\varepsilon + D_d K) y$$

$$= y^T D_d (-D_\varepsilon + K) y$$

$$= \frac{1}{2} y^T \left\{ (-D_\varepsilon + K)^T D_d + D_d (-D_\varepsilon + K) \right\} y$$

Sufficient conditions for stability $\Rightarrow$ existence of diagonal matrix $D_d$ which is positive definite as the solution to

$$(-D_\varepsilon + K)^T D_d + D_d (-D_\varepsilon + K) < 0$$

$\downarrow$

stability