Nonlinear Systems

Lecture 21

04/16/13

Last time

Input output stability

Finite gain $L^p$ stability

$$\|y\|_p \leq K \|u\|_p + \beta$$

$$V(x) \leq 0 \quad \forall T \geq 0$$

$$\dot{y} = \begin{cases} \dot{y}(t) & 0 \leq t < T \\ 0 & 0.10. \end{cases}$$

A state-space condition for $L^2$ stability of

$$\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*}$$

If there is a continuously differentiable, positive definite $V(x)$

st.
\[
\frac{\partial V}{\partial x} f(x) + \frac{1}{2\alpha} \frac{\partial V}{\partial x} g(x)^T g(x) \frac{\partial V}{\partial x} + \frac{1}{2} h^T(x) h(x) \leq 0 \text{ (HJ)}
\]

then \( \bar{\gamma} \) has an \( L_2 \) gain \( \leq \gamma \)

Hamilton-Jacobi inequality

Prof: let (HJ) hold

\[
V(x) = \frac{\partial V}{\partial x} x = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)^T u
\]

\[
V \leq \frac{\partial V}{\partial x} f(x) + \frac{1}{2\alpha} \frac{\partial V}{\partial x} g(x)^T g(x) \frac{\partial V}{\partial x} + \frac{\alpha}{2} u^T u
\]

\[
\left[ \begin{array}{c}
\alpha \cdot b \\
\frac{1}{2\alpha} a^2 + \frac{\alpha}{2} b^2
\end{array} \right] \iff 0 \leq \left( \frac{a}{\sqrt{\alpha}} - \sqrt{\alpha} b \right)^2
\]

Now from (HJ) with \( \alpha = \gamma^2 \)

\[
\Rightarrow \quad \dot{V} \leq -\frac{1}{2} h^T(x) h(x) + \frac{\gamma^2}{2} u^T u
\]

\[
= \frac{1}{2} y^T(t) y(t) + \frac{\gamma^2}{2} u(t)^2 u(t)
\]
Now, integrate from 0 to T:

\[ V(x(T)) - V(x(0)) \leq -\frac{1}{2} \| y_r \|^2 + \frac{\beta^2}{2} \| u_r \|^2 \]

\[ \downarrow \]

\[ -V(x(0)) \leq V(x(T)) - V(x(0)) \]

\[ \downarrow \]

\[ \| y_r \|^2 \leq \frac{\beta^2}{2} \| u_r \|^2 + 2 V(x(0)) \]

we have: \( \sqrt{a^2 + b^2} \leq |a| + |b| \)

\[ \Rightarrow \| y_r \|^2 \leq \frac{\beta^2}{2} \| u_r \|^2 + \sqrt{2 V(x(0))} \]

\[ \downarrow \]

\[ K \]

\[ \beta \]

Note:
Lyapunov-like functions that are used to establish input-output stability are known as "storage functions."
For linear systems

\[ x = Ax + Bu \]
\[ y = Cx \]

(HJ) holds with

\[ V(n) = \frac{1}{2} x^T P x \]

and it simplifies to

\[ x^T (A^T P + PA + \frac{1}{\delta^2} PBB^T P + \frac{1}{\delta} C^T C) x \leq 0 \]

\[ \downarrow \]

\[ A^T P + PA + \frac{1}{\delta^2} PBB^T P + \frac{1}{\delta} C^T C \leq 0 \]

\[ \text{negative definite} \]
* Bounded Real Lemma:

Suppose \( A \) is Hurwitz \( (\lambda_1(A) < 0 \text{ for } i = 1, \ldots, n) \)
and let \( \gamma^* \) denote the \( L_2 \)-induced gain of

\[
(\text{H}_\infty \text{ norm}) \rightarrow \text{peak value of Bode-mag. plot}
\]

\[
\begin{align*}
\mathbf{x} &= A\mathbf{x} + B\mathbf{u} \\
\mathbf{y} &= C\mathbf{x}
\end{align*}
\]

Then for every \( \gamma > \gamma^* \), there is \( P = P^T > 0 \) s.t.

\[
A^TP + PA + \frac{1}{\gamma^2}PB\tilde{B}^TP + CC < 0
\]

* Small gain thm:

\[
\begin{align*}
\mathbf{u}_1 &\quad \rightarrow e_1 \quad \rightarrow H_1 \quad \rightarrow d_1 \\
\mathbf{y}_2 &\quad \leftarrow H \quad \leftarrow e_2 \quad \leftarrow u_2 \\
H : [\mathbf{u}_1, \mathbf{u}_2] &\quad \rightarrow [\mathbf{y}_1, \mathbf{y}_2]
\end{align*}
\]
Suppose $H_i$ has lp gain $< k_i$, $\forall i = 1, 2, \ldots$

If $\| y_i \|_2 < 1$ then the fbk interconnection $H$ is

lp stable.

Proof,

$H_1 : \| y_i \|_p \leq k_i \| e_{1T} \|_p + \beta_i$

$H_2 : \| y_i \|_p \leq k_i \| e_{2T} \|_p + \beta_i$

\[ e_1 = u_1 + y_1 \]

\[ e_2 = u_2 + y_2 \]

This is a very general context; holds for everything

$\| y_{1T} \|_p \leq k_i \| u_{1T} + y_{2T} \|_p + \beta_i$

$\leq \delta_i \| u_{1T} \|_p + k_i \| y_{2T} \|_p + \beta_i$

$\leq \delta_i \| u_{1T} \|_p + k_i \| y_{2T} \|_p + \delta_i \| y_{2T} \|_p + \delta_i \beta_2 + \beta_i$
\[ \|y_1\|_p \leq \frac{\delta_1}{1-\delta_1 \delta_2} \|u_1\|_p + \frac{\delta_2}{1-\delta_1 \delta_2} \|u_2\|_p + \frac{\beta_1 + \delta_1 \beta_2}{1-\delta_1 \delta_2} \]

and similarly for \( \|y_2\|_p \)

\[ \delta_1 \delta_2 \neq 1 \quad \text{if} \quad \delta_1 \delta_2 = 1 \quad \text{well posedness is violated} \]
\[ \delta_1 \delta_2 > 1 \quad \text{positiveness of } L_0 \text{ gains is violated} \]

This is a sufficient condition on the stability of such interconnected systems, but it is highly conservative. Info. from phase is not taken into account!

Robust Control

\[ \Delta \]

\[ \text{\Delta can be anything as long as it is norm bounded} \]

In adaptive control the structure of uncertainty \( \Delta \) is taken into account.
$\Delta$: modeling uncertainty with $L_2$ gain $\leq \frac{1}{\bar{\delta}}$

$P$: Plant

If $L_2$ gain $\delta_P \leq \frac{1}{\bar{\delta}}$

$\Rightarrow$ robust stability

What does $L_p$ gain $\leq \gamma$ mean for a memoryless function?

\[ u \rightarrow h(u) \rightarrow y \]

\[ y = h(u) \]

\[ |y| = |h(u)| \leq K |u| , \quad K > 0 \]

-bow-tie!
Passivity

\[ u \xrightarrow{H} y \]

(Same \# of inputs and outputs)

\[ H : L_2e \rightarrow L_2e \text{ is passive if for all } u \in L_2e \]

and \( T > 0 \)

\[ \langle y_T, u_T \rangle_2 = \int_0^T y_T(t)u_T(t) \, dt \geq -\beta \]

for some \( \beta > 0 \).

If you don't have any initial condition \( \beta = 0 \).

i.e., \( \beta \) accounts for initial conditions and if \( x(0) = 0 \)

\[ \Rightarrow \langle y_T, u_T \rangle_2 > 0 \]