Lecture 17

Non linear Systems 03/28/13

Last time:

a) $\dot{x} = f(x,t)$

$$W_1(x) \leq V(x,t) \leq W_2(x)$$

$$\partial V/\partial t + \frac{\partial V}{\partial x} f \leq -W_3(x)$$

W_1, W_2 pos. definite, radially unbounded

W_3 pos. semidefinite & cts diff.ble

$f(x,t)$: Lipschitz in x bounded in t

$$\Downarrow$$

uniform stability $\oplus$ $W_3(x(t)) \xrightarrow{t\to\infty} 0$

b) $\dot{x} = A(t)x$

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -C^T(t)\underbrace{C(t)}_{\text{fat}}$$

$K_1 I \leq P(t) \leq K_2 I$

uniform observability $\Rightarrow$ uniform asymptotic stability

$\exists\ \alpha > 0,\ \delta > 0$

$$\int_{t_0}^{t_0+\delta} \Phi^T(\tau, t_0)\, C^T(\tau)\, C(\tau)\, \Phi(\tau, t_0) \geq \alpha I$$

for all t_0

Gradient algorithm for estimation of parameters

$$\underbrace{y(t)}_{\text{measured output}} = \underbrace{\psi^T(t)}_{\text{regressor}}\, \theta$$

θ → vector of unknown (constant parameters)

$$\tilde{\theta}(t) = \theta - \hat{\theta}(t)$$

$\tilde{\theta}(t)$ → estimation error

$\hat{\theta}(t)$ → estimate

Want to check conditions on $\psi(t)$ s.t.

$$\hat{\theta}(t) \xrightarrow{t \to \infty} \theta$$

$$\Updownarrow$$

$$\tilde{\theta}(t) \xrightarrow{t \to \infty} 0$$

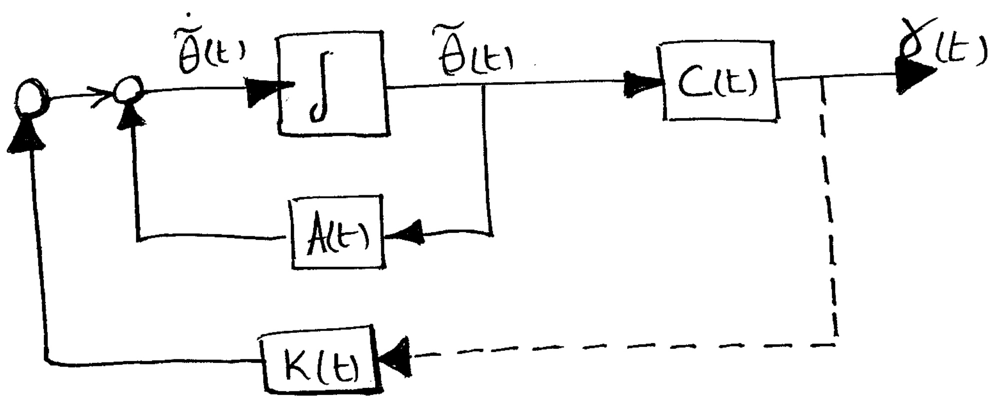
$$\dot{\tilde{\theta}}(t) = \underbrace{-\psi(t)\,\psi^T(t)}_{A(t)}\,\theta(t)$$

Fact : Uniform observability of $(A(t), C(t))$

$$\Updownarrow$$

Uniform obs. of $(A(t) + K(t)C(t),\ C(t))$

for bounded $K(t)$.



$$P(t) = \tfrac{1}{2} I \quad \Rightarrow \quad C^T(t)\,C(t) = \psi(t)\,\psi^T(t)$$

$$C(t) = \psi^T(t)$$

$$\boxed{A(t) + K(t)C(t)} = -\psi(t)\psi^T(t) + K(t)\,\psi^T(t) \overset{\text{if we select } K(t)=\psi(t)}{=\!=} \boxed{0}$$

state transition matix $= I$

$$\Downarrow$$

uniform obs. amounts to checking:

$$\int_{t}^{t_0+\delta} \psi(t)\psi^T(t)\,dt \geq \alpha I$$

(If satisfied $\Rightarrow$ $\psi(t)$ persistently exciting) (PE)

Ex1) $\psi(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\psi(t)\psi^T(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{not PE}$$

Ex2) $\psi(t) = \begin{bmatrix} 1 \\ \sin t \end{bmatrix}$

$$\psi(t)\psi^T(t) = \begin{bmatrix} 1 & \sin t \\ \sin t & \sin^2 t \end{bmatrix} = \begin{bmatrix} 1 & \sin t \\ \sin t & \frac{1-\cos 2t}{2} \end{bmatrix}$$

$\delta = 2\pi$

with any t_0 :

$$\int_{t_0}^{t_0+\delta} \psi(t)\psi^T(t)\,dt = \begin{bmatrix} 2\pi & 0 \\ 0 & \pi \end{bmatrix} \geq \alpha I$$

$$\Downarrow$$

PE

Ex 1b) $\psi(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \psi(t)\psi^T(t)$ rank 1 matrix

$\Rightarrow$ not uniformly observable

Ex 3) $\psi(t) = e^{-t}$

$$\int_{t_0}^{t_0+\delta} e^{-2t}\, dt > 0$$

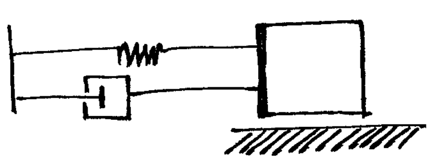
but no lower bound can be determined independently of t_0, therefore it is not uniformly observable (not PE)

Q: How to bring $\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t)$

$a_n = 1 \; ; \; n > m$

Into the form $y(t) = \psi^T(t)\theta$

Ex



$$m\ddot{y} + d\dot{y} + ky = u$$

$$\ddot{y} + \underbrace{\frac{d}{m}}_{a_1}\dot{y} + \underbrace{\frac{k}{m}}_{a_0} y = \underbrace{\frac{1}{m}}_{b_0} u$$

$$s^n Y(s) = -a_{n-1} s^{n-1} Y(s) - \cdots - a_0 Y(s) +$$
$$+ b_m s^m U(s) + \cdots + b_0 U(s)$$

→ Naive approach: multiply by $\frac{1}{s^n}$ and proceed... this is <u>bad</u> from the practicle point of view because integrators are marginally stable and proceeding with such an approach can lead to instability.

→ Instead multiply with

$$\frac{1}{\Lambda(s)} \; ; \; \Lambda(s) = s^n + \lambda_{n-1} s^{n-1} + \cdots + \lambda_1 s + \lambda_0$$

("stable polynomial")

LHS:

$$\frac{s^n}{\Lambda(s)} Y(s) = \left(1 - \frac{\lambda_{n-1} s^{n-1} + \cdots + \lambda_0}{\Lambda(s)}\right) Y(s)$$

$$= Y(s) - \frac{\lambda_{n-1} s^{n-1} + \cdots + \lambda_0}{\Lambda(s)} Y(s)$$

Ex : $\ddot{y} + a_1 \dot{y} + a_0 y = b_1 u$

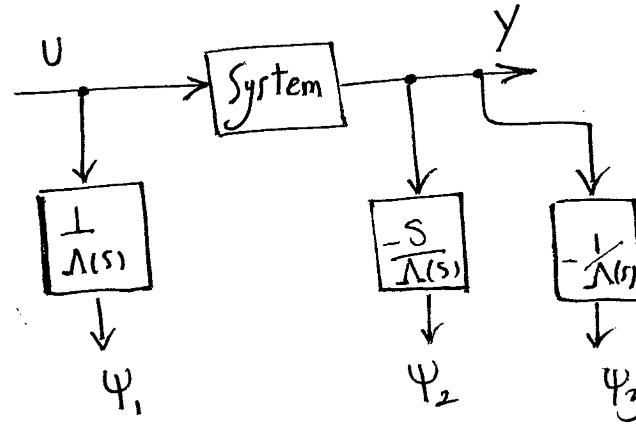
$$s^2 Y = -a_1 s Y - a_0 Y + b_1 U \Big/ \frac{1}{s^2 + \lambda_1 s + \lambda_0}$$

$$Y(s) = b_1 \frac{1}{\Lambda(s)} U(s) \underbrace{- (a_1 - \lambda_1) \frac{s}{\Lambda(s)} Y(s)}_{} - (a_0 - \lambda_0) \frac{1}{\Lambda(s)} Y(s)$$

you are measuring Y and passing it through a stable filter therefore everything is fine now.

$$\theta := \begin{bmatrix} b_1 \\ a_1 - \lambda_1 \\ a_0 - \lambda_0 \end{bmatrix} = \begin{bmatrix} b_1 \\ \bar{a}_1 \\ \bar{a}_0 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\psi(t) = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$



In order to get rich enough input to achieve P.E. you can shake the system with 2 sin functions (ex. $\sin(t) + \sin(10t)$)

Rule of thumb : one sin. per 2 input parameters.

this is just a rule of thumb and if this does not achieve uniform observability you can add another sin.

Ex (from last time) but more general

$$\dot{x}_1 = -a x_1 + W^T(t)\, x_2(t) \quad : \quad \text{scalar state } x_1$$
$$\dot{x}_2 = -W(t)\, x_1 \quad : \quad x_2(t) \in \mathbb{R}^{n-1} \qquad \ldots (1)$$

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last time : $a=1$ $W(t) \in \mathbb{R}$

Uniform stability ⊕ $x_1(t) \xrightarrow{t \to \infty} 0$

Q: conditions on $W(t)$ st. (1) is unif. asymptotically stable.

$$A(t) = \begin{bmatrix} -a & W^T(t) \\ -W(t) & 0 \end{bmatrix}$$

With $P = \frac{1}{2a} I$

$$Q = C^T C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C}$$

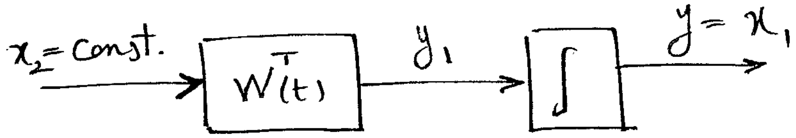
$$A(t) + K(t)C = \begin{bmatrix} -a & W^T(t) \\ -W(t) & 0 \end{bmatrix} + \begin{bmatrix} K_1(t) & 0 \\ K_2(t) & 0 \end{bmatrix} \Bigg|_{\substack{K_1(t) = a \\ K_2(t) = W(t)}}$$

$$= \begin{bmatrix} 0 & W^T(t) \\ 0 & 0 \end{bmatrix}$$

$$\dot{x}_1 = W^T(t)\, x_2$$

$$\dot{x}_2 = 0 \qquad \ldots (2)$$

$$y = x_1$$



you can show that uniform observability is preserved through integration, i.e.,

Unif observability (2) $\Leftrightarrow$

$$\dot{x}_2 = 0$$

$$y_1 = W^T(t)\, x_2(t)$$

$$\Rightarrow \quad \int_{t_0}^{t_0+\delta} W(t)\, W^T(t)\, dt \geq \alpha I$$

W: P.E. $\Rightarrow$ (1) Uniformly asymptotically stable

Key: Passing output through an integrator doesn't change observability!