# Nonlinear Systems

## Lecture 16 03/26/13

Stability of time varying systems
(Uniform)

Lyapunov based characterization

$$W_1(x) \leq V(x,t) \leq W_2(x) \qquad (1)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x) \leq -W_3(x) \qquad (2)$$

Question: What happens when $W_3$ is only positive semi-definite?

Note! LaSalle's invariance principle doesn't hold, but the following is true:

**Thm** Suppose (1) and (2) hold with:

- $W_1, W_2$ : positive definite
- $W_3$ : positive semi-definite

Suppose further that $W_1$ is radially unbounded and $f(t,x)$ is locally Lipschitz (in $x$) and bounded in $t$.

Then $W_3(x(t)) \xrightarrow{t \to \infty} 0$.

Note! Much weaker than invariance principle (convergence to the largest invariant set)

In time invariant case – we had as an example for LaSalle's

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin x_1 - x_2$$

$$\dot{V} = -x_2^2 = 0 \Rightarrow x_2 \equiv 0 \qquad \left( \dot{x}_2 \equiv 0 = -\sin x_1 - 0 = 0 \Rightarrow x_1 = 0 \right)$$

$(x_1, x_2) = (0,0)$ : largest invariant set

Ex

$$\dot{x}_1 = -x_1 + w(t) x_2$$
$$\dot{x}_2 = -w(t) x_1$$

$$V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \Rightarrow \dot{V} = -x_1^2 + \cancel{x_1 w(t) x_2} - \cancel{x_2 w(t) x_1}$$

$$W_3(x) = x_1^2 + 0 x_2^2 \geq 0$$

$$W_3(x) = x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x$$

Conclusion:

Uniform Stability $\oplus$ $x_1(t) \xrightarrow{t\to\infty} 0$

Note! Later - conditions on $W(t)$ that allow us to conclude uniform asymptotic stability.

Key result that allow us to prove this Thm

<u>Barbalat's Lemma</u>: $\int_0^\infty \phi(t)\,dt < \infty$

[Translation $\lim_{T\to\infty} \int_0^T \phi(t)\,dt$ exists and is bounded]

$\oplus$

Uniform continuity $|t_1 - t_2| < \delta \Rightarrow |\phi(t_1) - \phi(t_2)| < \varepsilon$

$\delta, \varepsilon$ independent on $t_1$ & $t_2$

$\Downarrow$

$$\lim_{t\to\infty} \phi(t) = 0$$

Note! 1. If $\dot{\phi}(t)$ is bounded $\Rightarrow \phi$ is uniformly cts.

2. Uniform continuity CANNOT be relaxed.

Ex $\phi(t)$ sequence of pulses centered @ $K=1,2,3,\ldots$ of amplitude $K$ and width $1/K^3$.

$$\int_0^\infty \phi(t)\,dt = \sum_{K=1}^{\infty} \frac{1}{K^2} < +\infty \qquad \left(\text{but } \lim_{t\to\infty} \phi(t) \neq 0\right)$$

Linear Systems $\dot{x} = A(t)\,x$

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t)$$

$$K_1 I \leq P(t) \leq K_2 I$$

Let $Q(t) = C^T(t)\underbrace{C(t)}_{\text{fat matrix}} \geq 0$

Aside
$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - bx_2 \end{aligned}\right\} \quad \begin{aligned} V(x) &= \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 \\ \dot{V}(x) &= -bx_2^2 = -[x_1 \ x_2]\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}}_{Q}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

$$Q = C^T C = \begin{bmatrix} 0 \\ \sqrt{b} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{b} \end{bmatrix}$$

Observability matrix $\quad W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{b} \\ -\sqrt{b} & * \end{bmatrix}$

Thm. If $(A(t), C(t))$ is uniformly observable then $\bar{x} = 0$ is uniformly asymptotically (exponentially) stable.

If $\dot{x} = A(t)x \quad ; \quad V = x^T P(x) x$

[redacted]

$$\dot{V} = -C^T(t) C(t)$$

Def. The pair $(A(t), C(t))$ is unif. observable if there exists $\alpha > 0$ and $\delta > 0$ such that,

$$\int_{t_0}^{t_0+\delta} \Phi^T(\tau, t_0)\, C^T(\tau)\, C(\tau) \underbrace{\Phi(\tau, t_0)}_{\text{state transition matrix of } A(t)} d\tau \geq \alpha I$$

$\alpha I$ → scaled identity matrix

for all $t_0$.

in time invar. case we had $\int e^{A^T(\tau - t_0)} C^T C\, e^{A(\tau - t_0)} d\tau$

$$\begin{cases} \dfrac{\partial \Phi(t,t_0)}{\partial t} = A(t)\, \Phi(t,t_0) \\[2ex] \Phi(t_0,t_0) = I \end{cases}$$

**Ex** Gradient Algorithm for Estimation of Parameters (identification)

$y(t)$ : Scalar output ; $y(t) \in \mathbb{R}$

$$y(t) = \underbrace{\varphi^T(t)}_{\text{"regressor", vector of time dependent function}}\ \theta \longrightarrow \text{vector of constant but unknown parameters}$$

$$\begin{matrix} \theta \in \mathbb{R}^p \\ \varphi(t) \in \mathbb{R}^p \end{matrix} \qquad (1)$$

Note! linear dependence on unknown parameters

for example covers

$$\sum_{i=0}^{n} a_i y^{(i)}(t) = \sum_{i=0}^{m} b_i u^{(i)}(t) \; ; \; \begin{matrix} a_n = 1 \\ n > m \end{matrix} \qquad (2)$$

Next time we'll show how to go from (2) to (1).

Objective: estimate unknown parameters.

$\hat{\theta}(t)$ : vector of parameter estimates

estimation error $\quad \tilde{\theta}(t) = \underbrace{\theta}_{\text{constant but unknown}} - \hat{\theta}(t)$

$$\dot{\tilde{\theta}}(t) = -\dot{\hat{\theta}}(t)$$

$$\boxed{e(t) =} \underbrace{y(t)}_{\text{real output}} - \underbrace{\hat{y}(t)}_{\text{estimated output}}$$

$$= \psi^T(t)\,\theta - \psi^T(t)\,\hat{\theta}(t)$$

$$\boxed{= \psi^T(t).\tilde{\theta}(t)} \qquad \text{(E)}$$

cost function:

$$J = \tfrac{1}{2} e^2(t) = \tfrac{1}{2} \underbrace{\tilde{\theta}^T(t)\,\psi(t)}_{e^T(t)} \underbrace{\psi^T(t)\,\tilde{\theta}(t)}_{e(t)}$$

Gradient - based algorithm:

$$\dot{\tilde{\theta}}(t) = -\frac{\partial J}{\partial \tilde{\theta}}$$

$$\dot{\tilde{\theta}}(t) = -\psi(t)\psi^T(t)\,\tilde{\theta}(t)$$

linear differential equation

$$y(t) = \psi(t)\,\theta$$

$$\dot{\tilde{\theta}}(t) = A(t)\,\tilde{\theta}(t) \qquad \tilde{\theta}(0) = \theta - \hat{\theta}(0)$$

$$A(t) = -\psi(t)\psi^T(t)$$

So in order to study the behavior of $\tilde{\theta}$ and if our estimate converges to $\theta$, we must study this linear differential equation.

Since we don't know $\theta$ and $\tilde{\theta}(0) = \theta - \hat{\theta}(0)$ we can't simulate the behavior of $\tilde{\theta}$ but we can analyze it.

$$\dot{\hat{\theta}}(t) = -\dot{\tilde{\theta}}(t) = +\psi(t)\psi^T(t)\,\tilde{\theta}(t) \overset{E}{\equiv} \psi(t)\,e(t)$$

$\hat{\theta}(0)$ : our choice

$\psi(t)$ : known regressor vector

$e(t) = y(t) - \psi^T(t)\hat{\theta}(t)$

$$\dot{\hat{\theta}}(t) = -\psi(t)\psi^T(t)\hat{\theta}(t) + \psi(t)y(t)$$

measured output

Summary:

Convergence of parameters depends on properties of:

$$\dot{\tilde{\theta}}(t) = A(t)\,\theta(t)$$

$$A(t) = -\psi(t)\psi^T(t)$$

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$$\int_{t_0}^{t_0+\delta} \Phi^T(\tau, t_0)\, C^T(\tau)\, C(\tau)\, \Phi(\tau, t_0)\, d\tau \geq \alpha I$$

* If $P = \frac{1}{2} I \Rightarrow Q(t) = A(t) = \psi(t)\psi^T(t) = C^T(t)C(t)$

$$\Rightarrow C(t) = \psi^T(t)$$

If $(A(t), C(t))$ is uniformly observable, so is $(A(t) + K(t)C(t), C(t))$ with a bounded $K(t)$, i.e, observability is preserved after feedback.