Nonlinear Systems

Lecture 11 02/26/13

Last time:

- Lyapunov — stability
  — functions (i.e. indirect method)

main result: $\dot{x} = f(x)$; $f(0) = 0$

$V: D \rightarrow \mathbb{R}$ positive definite
$(V(0) = 0 \ , \ V(x) > 0 \text{ for all } x \in D)$

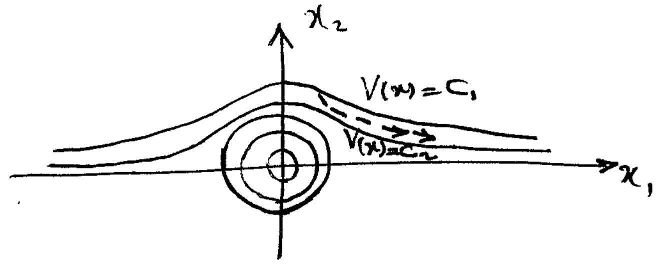
1. $\dot{V}(x) \leq 0$, for all $x \in D \Rightarrow \bar{x} = 0$ is stable
(negative semidefinite)

2. $\dot{V}(x) < 0$ for all $x \in D \setminus \{0\} \Rightarrow \bar{x} = 0$ is LAS
$\dot{V}(0) = 0$

GAS: V: globally positive definite
radially unbounded
$(\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty)$
+
$\dot{V}$: globally negative definite

$\Omega_c = \{x ; V(x) \leq c\}$

$$V(x) = \frac{x_1^2}{x_1^2+1} + x_2^2 \qquad c_1 < c_2$$

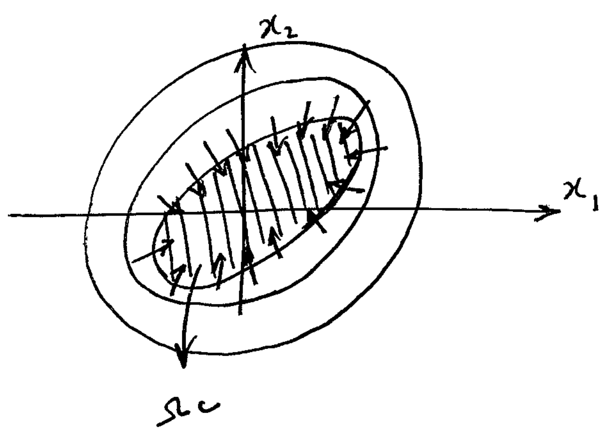


this example shows why radial unboundedness is crucial for Global AS.

1. Stability follows from positive invariance of Ω_c

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \nabla V(x) . f(x) \leq 0$$

$x_0 \in \Omega_c \Rightarrow x(t) \in \Omega_c$ for all t

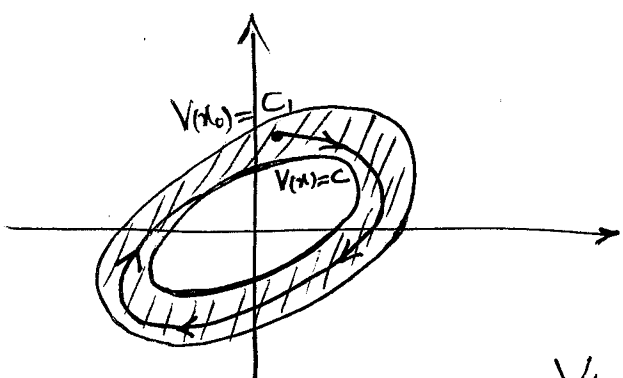


2. (Sketch of the proof) $\dot{V}(x) < 0 \quad , \forall x \in D \setminus \{0\}$

$\frac{dV}{dt} < 0 \Rightarrow$ V(t) is decreasing function bounded from below by zero

$\Rightarrow \quad \exists\, c$ s.t. $\lim_{t\to\infty} V(t) = c$

assume $c>0$ and show that this is not possible.



$\max \dot{V} = -\gamma < 0$

$\{x ; c < V(x) < c_1\}$

$$V(x(t)) = V(x_0) + \int_0^t \underbrace{\dot{V}(x(\tau))}_{-\gamma}\, d\tau \leq$$

$$\leq V(x_0) - \gamma t$$

$\Rightarrow$ For t large enough $V(x(t))$ will become negative

$\Downarrow$

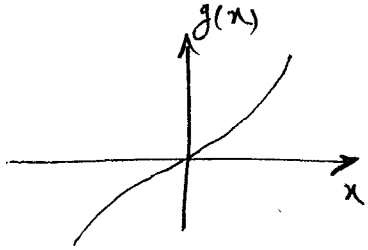
contradiction

Thus c has to be zero.

Examples

1) $\dot{x} = -g(x) \quad ; \quad x(t) \in \mathbb{R}$

1a)



$g(0) = 0$

$x \cdot g(x) > 0, \ \forall x$

e.g. $\quad g(x) = \underset{\downarrow K}{x} \quad ; \Rightarrow g(0) = 0$

$x \cdot g(x) = \underset{\downarrow K}{x^2} > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

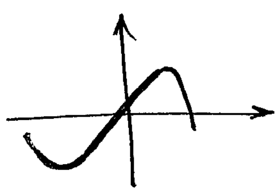
$\underline{K > 0}$

e.g. $\quad g(x) = x^{2n+1} \quad n \in \mathbb{N} = \{1, 2, 3, \dots\}$

$\Rightarrow g(0) = 0$

$x \cdot g(x) = x^2 \cdot x^{2n} > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$

1b)



$g(0) = 0$

$x g(x) > 0 \quad \forall x \in (-b, c) \setminus \{0\}$

or $(-a, a)$

e.g. $\quad g(x) = \sin(x) \quad ; b = c = \pi$

$g(x) = x - x^3 \quad ; b = c = 1$

* How to choose Lyapunov functions?

start with $\quad V(x) = \frac{1}{2} x^2$ → g.p.d. (globally positive definite) → radially unbounded

$$\dot{V}(x) = \underbrace{\frac{1}{2} 2x}_{\frac{\partial V}{\partial x}} \cdot \underbrace{\dot{x}}_{f(x)} = x(-g(x)) = -x g(x) < 0$$

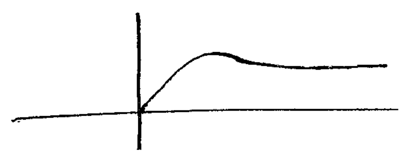
for all $x \in D \setminus \{0\}$

$$D = \begin{cases} \mathbb{R} \text{ in (a)} \Rightarrow \text{GAS} & \text{(Global)} \\ (-b,c) \text{ in (b)} \Rightarrow \text{LAS} & \text{(Local)} \end{cases}$$

Alternative choice:

$$V(x) = \int_0^x g(\xi)\,d\xi \qquad \left(\text{Note! if } g(x) = x \Rightarrow V(x) = x^2\right)$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g(x) \cdot (-g(x)) = -g^2(x)$$

for 1(a) as long as we have 
we will have radial unboundedness

$$\forall x \in \begin{cases} \mathbb{R} \setminus \{0\} \text{ in (a)} \Rightarrow \text{GAS} \\ (-b, c) \text{ in (b)} \Rightarrow \text{LAS} \end{cases}$$

2) $\dot{x}_1 = x_2$

$\dot{x}_2 = -a g(x) - b x_2$

Pendulum special case of this with $g(x) = \sin x$

Last time: $V(x) = a \int_0^{x_1} g(\xi)\,d\xi + \frac{1}{2} x_2^2$

$$\dot{V} = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[a g(x_1) \quad x_2\right] \begin{bmatrix} x_2 \\ -a g(x_1) - b x_2 \end{bmatrix}$$

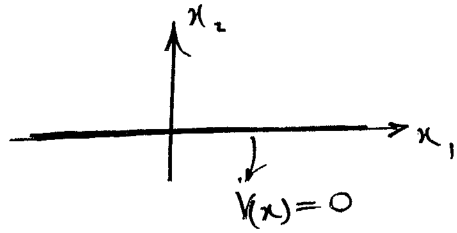
$$= a\cancel{x_2 g}(x_1) - a\cancel{x_2 g}(x) - b x_2^2$$

$$\Rightarrow \dot{V}(x) = -0\, x_1^2 - b x_2^2$$

$b = 0 \Rightarrow$ no damping (energy conserved)

$b > 0 \Rightarrow \dot{V}$ is negative semi-definite

choose $x_2 = 0$



$\dot{V}(x) \leq 0$ for all $x \in D \setminus \{0\}$

$$\Downarrow$$

From our main result $\Rightarrow$ we can only conclude stability in the sense of Lyapunov, not asymptotic stability.

We know that pendulum would eventually settle down (if $b > 0$)

↳ there is viscous damping

Q: How can we sort out this issue?

Two options:

- choose different Lyapunov function (later)
- La Salle's invariance principle

LaSalle's Invariance Principle

- applicable to time invariant systems
- can conclude asymptotic stability even when $\dot{V}(x) \leq 0$

Let $\Omega_c = \{ x;\ V(x) \leq c \}$ be bounded and let $\dot{V}(x) \leq 0$ in Ω_c.

Define $S := \{ x \in \Omega_c\ ;\ \dot{V}(x) = 0 \}$ and let M be the largest invariant set in S. Then for every $x(0) \in \Omega_c \Rightarrow x(t) \longrightarrow M$. $\blacksquare$

In our example

$$\dot{V}(x) = -b x_2^2 = 0 \Rightarrow x_2 = 0$$

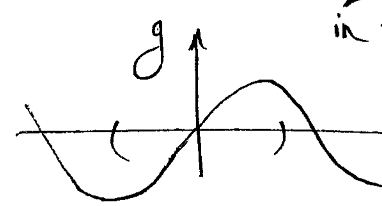
$$S = \{ x_1 \in (-b, c) \, , \, x_2 = 0 \}$$

$$x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0$$

$$\downarrow$$

$$a g(x_1) - b \cancelto{0}{x_2} \equiv 0$$

$g(x_1) \equiv 0 \Rightarrow$ (can only happen for $x_1 = 0$)



in the domain around the origin $x_1 = 0$ is the only point.