Nonlinear Systems

Lecture 05 02/05/13

Last time:

- Phase portraits of 2nd order systems
- Hartman-Grobman Thm
- Bendixon Thm (absence of periodic orbits in 2nd order systems)

Today:

- Application of Bendixon Thm
- Poincaré-Bendixon Thm (existence of periodic orbits in 2nd order systems)
- Hopf Bifurcation (super > critical)

Bendixon: \( \mathcal{D} \) simply connected domain (region w/o holes)

Dynamics:
\[
\begin{align*}
    \dot{x}_1 &= f_1(x_1, x_2) \\
    \dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]
\( x_i(t) \in \mathbb{R} \)

- dirf: not identically zero AND doesn't change sign in \( \mathcal{D} \)

\[ \mathcal{P} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \]

\( \Rightarrow \) No periodic orbits in \( \mathcal{D} \)
**Ex 1**  
Given $A \in \mathbb{R}^{2 \times 2}$ unless $\text{trace}(A) = 0 \Rightarrow$ no periodic orbits

\[
\begin{align*}
    \dot{x}_1 &= ax_1 + bx_2 \\
    \dot{x}_2 &= cx_1 + bx_2
\end{align*}
\]

$\iff A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; 
$\frac{\partial f_1}{\partial x_1} = a$ 
$\frac{\partial f_2}{\partial x_2} = d$ 
$\text{div} F = a + d = \text{trace}(A)$

**Note!** In general, GAS of a unique e.p. out rules the presence of any periodic orbit.

**Ex 2**

\[
A = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \quad \lambda_{1,2}(A) = \pm j\beta
\]

\[
\begin{align*}
    \dot{x}_1 &= x_2 = f_1 \\
    \dot{x}_2 &= -\alpha x_2 + x_1 - x_1^3 + x_2^2 = f_2
\end{align*}
\]

$\lambda_{1,2}(A) = \pm j\beta$

$\div F = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \alpha + x_1^2 \Rightarrow \div F = x_1^2 - \alpha$

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*Bendixson inconclusive*

*can't use Thm in this region (we don't know about periodic orbits)*
Bendixson can only tell you about the absence of periodic orbits. It cannot tell you if you have them, it can only rule their presence.

**Aside**

**Invariant Sets**

\[ x = f(x), \quad x(0) = x_0 \]

A trajectory starting at \( x_0 \) will be denoted by \( \phi(t, x_0) \).

A set \( M \) is positively (negatively) invariant if \( x_0 \in M \Rightarrow \phi(t, x_0) \in M, \quad \forall t > 0 \) (\( \forall t < 0 \))

So what condition should be satisfied for this to happen?

\( f(x) \) should always point into the set.

**Ex. 1** Predator-Prey model

\[ \dot{x} = (a-bx) x \quad \text{prey} \]

\[ \dot{y} = (c-d) y \quad \text{predator} \]

\( a, b, c, d \) positive constants, \( xy \): chance of encounter
\[ f_1(x,y) = ax \]
\[ f_2(x=0,y) = -dy \]

**Ex 2**
\[ \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) \]
\[ \dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \]

We'll show that, \( B_r = \{ x \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2 \} \)
is positively invariant for sufficiently large \( r \) (to be determined)

\[ V(x) = x_1^2 + x_2^2 = r^2 \]
\[ \nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \]
\[ \nabla V(x) = f_1 \frac{\partial V}{\partial x_1} + f_2 \frac{\partial V}{\partial x_2} \]
\[ = 2x_1(x_1+x_2-x_1(x_1^2+x_2^2)) + 2x_2(-2x_1+x_2-x_2(x_1^2+x_2^2)) \]
\[ = -2(x_1^2+x_2^2)^2 + 2x_1 + 2x_2 - 2x_1x_2 \]
\[ = -2(x_1^2+x_2^2)^2 + 2x_1 + 2x_2 - 2x_1x_2 \]
\[ \leq -2(x_1^2+x_2^2)^2 + 2(x_1^2+x_2^2) + x_1 + x_2 \]
\[ = -2(x_1^2+x_2^2)^2 + 3(x_1^2+x_2^2) \]
\[ = -2r^4 + 3r^2 \]
\[ = -2r^2(r^2 - 3/2) \]
\[ \\
\text{yes if } r^2 \geq 3/2 \text{ (or } r \geq \sqrt{3/2} \text{) } \]

So \( f(x) \cdot \nabla V(x) \leq 0 \) if \( r^2 \geq 3/2 \)

**Poincare-Bendixon Thm:**

**Given** 2nd order system: \( \dot{x} = f(x) \), \( x(t) \in \mathbb{R}^2 \)

M: compact (closed and bounded) set (connected set)

\( f \) a) there are no equilibrium points in M, and
   b) M is positively invariant

\( \Rightarrow M \) contains aperiodic orbit.
\[ \begin{align*}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= x_1
\end{align*} \]

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

Unique e.p. is the origin \( \bar{x} = (0, 0) \)

\[ M = \{ \mathbf{x} \in \mathbb{R}^2 \mid r^2 \leq x_1^2 + x_2^2 \leq R^2 \} \]

\[ \nabla V(\mathbf{x}) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \]

\[ \mathbf{f}(\mathbf{x}) \cdot \nabla V = (-x_2, x_1) \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \]

\[ = -2x_1x_2 + 2x_1x_2 = 0 \quad \text{(on the boundary of \( M \))} \]

\( M \) positively invariant \( \odot \) doesn't contain e.p.

\[ \Rightarrow \text{there is a periodic orbit in} \; M \]

(in fact, there are \( \infty \) many of them)

**Note!** It can be shown that the Thm also holds if \( M \) contains a single equilibrium point which is either an unstable node or unstable focus.
If you take out the e.p.

In example 2:
\[
\begin{align*}
    \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) = f_1 \\
    \dot{x}_2 &= 2x_1 + x_2 - x_2(x_1^2 + x_2^2) = f_2
\end{align*}
\]

\[
A = \left. \frac{df}{dx} \right|_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}
\]

unstable focus

Unstable focus (spiral)

Aside

Unstable node for n=2
\[
\lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 > 0, \quad \lambda_2 > 0
\]

\[
\begin{align*}
    \dot{z}_1 &= z_1 \\
    \dot{z}_2 &= 2z_2
\end{align*}
\]

or