Nonlinear Systems

Lecture 24  01/31/13

Last time

Transcritical } bifurcations \[ \dot{x} = \alpha x \pm x^2 \]

Pitchfork } \[ \dot{x} = \alpha x \pm x^3 \]

Intro to phase portraits of 2nd order systems

Today:
Finish previous discussions

Criteria for absence \hspace{1cm} \text{(Bendixson)}

or presence \hspace{1cm} \text{(Poincare-Bendixson)}
of periodic orbits \hspace{1cm} \text{(limit cycles or neutrally stable periodic of 2nd order systems orbits)}

\underline{a) Linear systems :}

\[ A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{R} \]

\[ \tilde{A} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \quad \lambda \in \mathbb{R} \]

\[ \bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad \lambda = \alpha \pm j\beta \]
Stable node \((\lambda_1, \lambda_2 \in \mathbb{R}; \lambda_1, \lambda_2 < 0)\)

\[ z_1(t) = z_1(0) e^{\lambda_1 t} \]

**Aside** (from last time)

\[ e^{\lambda_1 t} = \frac{z_1(t)}{z_1(0)} \]

\[ z_2(t) = z_2(0) \left( e^{\lambda_1 t} \right)^{\frac{\lambda_2}{\lambda_1}} = z_2(0) \left[ \frac{z_1(t)}{z_1(0)} \right]^{\frac{\lambda_2}{\lambda_1}} \]

**Unstable node** \(\lambda_1 > \lambda_2 > 0\)

saddle \(\lambda_2 < 0 < \lambda_1\)

These 2 directions are special because the corresponding e-vectors will be \((\text{s})\) and \((\text{b})\)
In the original coordinates: \( x = T^{-1} z \)

determined by eigenvectors of \( A \)

\[ z_1 = \alpha z_1 - \beta z_2 \]
\[ z_2 = \beta z_1 + \alpha z_2 \]

\[ \lambda_{1,2} = \alpha \pm j \beta \]

Complex-conjugate eigenvalues

Polar coordinates

\[ z_1 = r \cos \theta \]
\[ z_2 = r \sin \theta \]

In polar coordinates

\[ r = \alpha r \]
\[ \dot{\theta} = \beta \]
\[ \theta(t) = \theta_0 + \beta t \]
Phase portraits of nonlinear systems with hyperbolic equilibrium points (linearization around equilibrium doesn't have e-values on jω axis) can be related to phase portrait of corresponding linear systems via Hartman-Grobman Thm.

Cases that can be covered by this => saddle, node, focus

Hartman-Grobman Thm:

If $\bar{x}$ is a hyperbolic eq. point of $\dot{x}=f(x), x(t) \in \mathbb{R}^n$, then there is a homeomorphism from a neighborhood of $\bar{x}$ to $\mathbb{R}^n$ that maps trajectories of $\dot{x}=f(x)$ to those of corresponding linearizations.

homeomorphism: a cts map with cts inverse
Note: Absence of e-values on the imaginary axis is of essence.

Ex.

\[
\begin{align*}
\dot{x}_1 &= -x_2 + ax_1(x_1^2 + x_2^2) \\
\dot{x}_2 &= x_1 + ax_2(x_1^2 + x_2^2)
\end{align*}
\]

\text{polar coord.} \quad \begin{align*}
\dot{r} &= ar^3 \\
\dot{\theta} &= 1
\end{align*}

\[
\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Linearization around the origin

\[
A = \frac{df}{dx} \bigg|_{\bar{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Harmonic oscillator

For the nonlinear system:

\(a > 0\) unstable focus

\(a < 0\) stable focus

An example of a problem with e-values at the origin \((0)\)

\(\ddot{y} = 0\) (double integrator, e.g. puck on frictionless surface)

\[
\begin{align*}
\dot{x}_1 &= y \\
\dot{x}_2 &= \dot{y}
\end{align*}
\]

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

\[
x_1(t) = x_1(0) + x_2(0)t
\]

\[
x_2(t) = \text{const} = x_2(0)
\]
Bendixon criterion:

(Criterion for the absence of periodic orbits)

\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*} \]

\[ \text{div} f = \nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \]

If \( \text{div} f \) is not identically equal to zero and does not change sign in a simply connected region \( D \), then there are no periodic orbits in \( D \).

Simply connected domain \( \rightarrow \) region without any holes.

Proof: Assume there is a periodic orbit (closed traj.) in \( D \)
\[ \int_{S} f(x) \cdot n(x) \, dl = \int_{S} \nabla f(x) \cdot dx, \quad S \neq 0 \]

If \( \nabla f(x) \) does not equal zero or change sign the right hand side will never be zero. (\# contradiction)

So there cannot be such a periodic orbit in \( D \).

**Ex.**

\[ x = Ax \quad \text{div}_f = \text{trace}(A) \]

\[ A = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix} \quad \text{unless trace}(A) = 0 \Rightarrow \text{no periodic orbits} \]
FIGURE 2.5. Illustrating the Hartman Grobman theorem