Nonlinear Systems

Lecture 04 01/31/13

Last time

Transcritical } bifurcations $\dot{x} = \alpha x \pm x^2$

Pitchfork } $\dot{x} = \alpha x \pm x^3$

Intro to phase portraits of 2nd order systems

Today:

Finish previous discussions

Criteria for absence (Bendixon)

or presence (Poincare-Bendixon)

of periodic orbits (limit cycles or neutrally stable periodic orbits)

of 2nd order systems

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a) Linear systems:

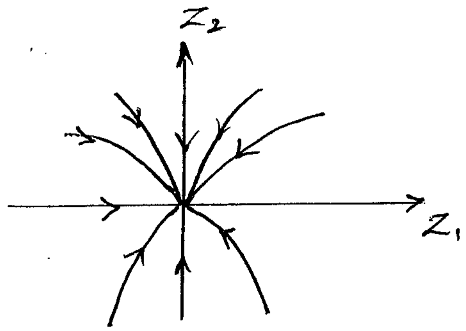
$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{array}{l} \lambda_1 \neq \lambda_2 \\ \lambda_1, \lambda_2 \in \mathbb{R} \end{array}$$

$$\bar{A} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \qquad \lambda \in \mathbb{R}$$

$$\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} ; \quad \lambda_{1,2} = \alpha \pm j\beta$$

Stable node $(\lambda_1, \lambda_2 \in \mathbb{R} \quad \lambda_1 < \lambda_2 < 0)$

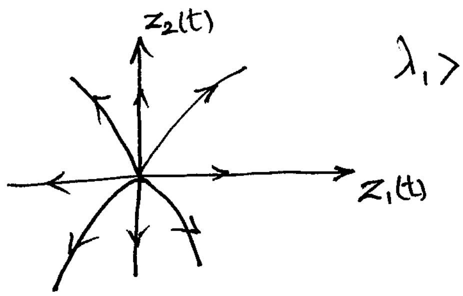
$$z_i(t) = z_i(0)e^{\lambda_i t}$$



<u>Aside</u> (from last time)

$$e^{\lambda_1 t} = \frac{z_1(t)}{z_1(0)}$$

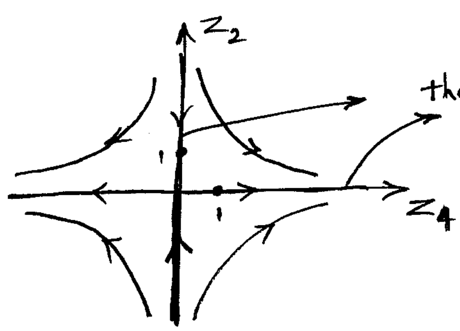
$$z_2(t) = z_2(0)\left(e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} = z_2(0)\left[\frac{z_1(t)}{z_1(0)}\right]^{\lambda_2/\lambda_1}$$



$\lambda_1 > \lambda_2 > 0$

unstable node

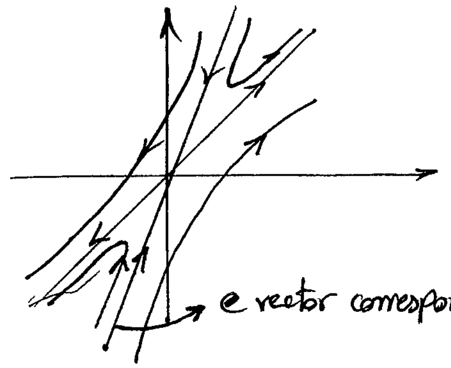
saddle $\quad \lambda_2 < 0 < \lambda_1$



these 2 directions are special because the corresponding e-vectors will be $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$

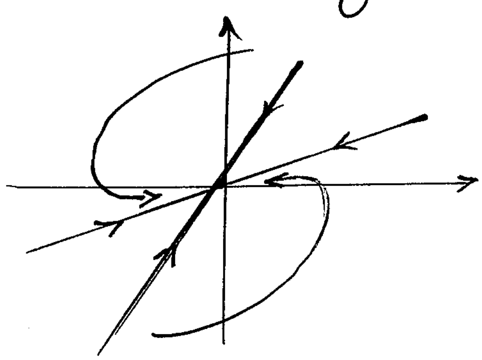
In the original coordinates : $x = T^{-1} z$

determined by e-vectors of A



e vector corresponding to $\lambda_2 < 0$

stable node in original coordinates



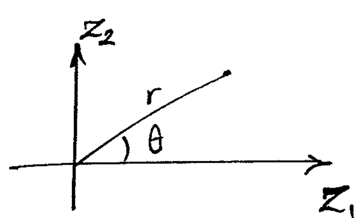
Complex - conjugate e-values

$$\left.\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \beta z_1 + \alpha z_2 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \alpha \pm j\beta$$

Polar coordinates

$z_1 = r\cos\theta$

$z_2 = r\sin\theta$

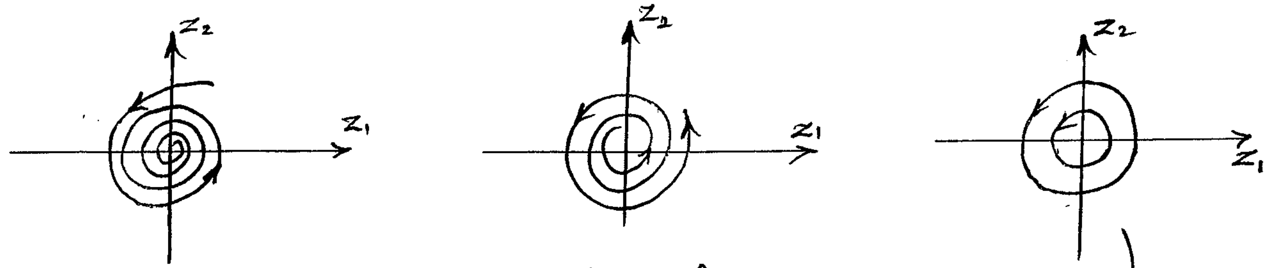


In polar coord.

$\dot{r} = \alpha r$

$\dot{\theta} = \beta \qquad \theta(t) = \theta_0 + \beta t$

a) $\alpha < 0$ b) $\alpha > 0$ c) $\alpha = 0$



stable focus unstable focus

not limit cycle
not an isolated closed traj.

Phase portraits of nonlinear systems with hyperbolic equilibrium points (linearization around equilibrium doesn't have e-values on $j\omega$ axis) can be related to phase portrait of corresponding linear systems via Hartman-Grobman Thm.

Cases that can be covered by this $\Rightarrow$ saddle, node, focus

Hartman-Grobman Thm:

If $\bar{x}$ is a hyperbolic eq. point of $\dot{x} = f(x)$, $x(t) \in \mathbb{R}^n$, then there is a homeomorphism from a neighborhood of $\bar{x}$ to $\mathbb{R}^n$ that maps trajectories of $\dot{x} = f(x)$ to those of corresponding linearizations.

homeomorphism: a cts map with cts inverse

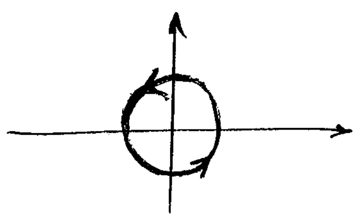
Note: Absence of e-values on the imaginary axis is of essence.

Ex.

$$\left.\begin{aligned} \dot{x}_1 &= -x_2 + a x_1 (x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + a x_2 (x_1^2 + x_2^2) \end{aligned}\right\} \xrightarrow[\text{coord.}]{\text{Polar}} \begin{aligned} \dot{r} &= a r^3 \\ \dot{\theta} &= 1 \end{aligned}$$

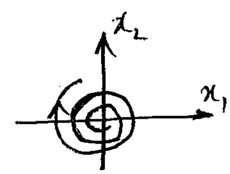
$$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

linearization around the origin $\quad A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

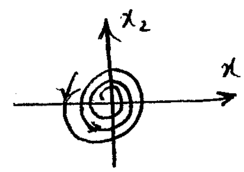


Harmonic oscillator

for the nonlinear system: $\quad a > 0$ unstable focus



$a < 0$ stable focus

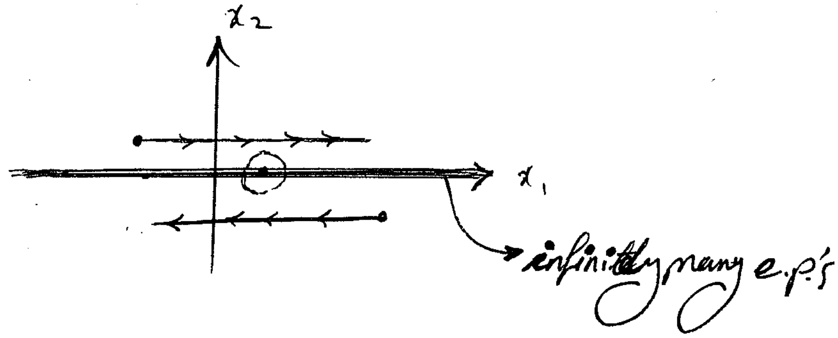


An example of a problem with e-values at the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\ddot{y} = 0 \quad$ (double integrator, e.g. puck on frictionless surface)

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x_1(t) &= x_1(0) + x_2(0)t \\ x_2(t) &= \text{const} = x_2(0) \end{aligned}$$



Bendixen criterion:

(Criterion for the absence of periodic orbits — limit cycles (Vanderpol); neutrally stable cycles (pendulum))

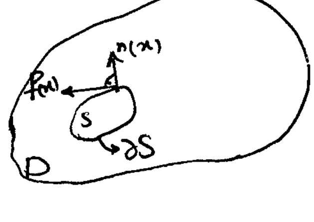
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f(x_1, x_2)$$

$$\operatorname{div} f = \nabla . f = \left[\partial/\partial x_1 \quad \partial/\partial x_2 \right] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

If div(f) is not identically equal to zero and does not change sign in a simply connected region D, then there are no periodic orbits in D.

Simply connected domain $\longrightarrow$ region without any holes. ○

Proof: Assume there is a periodic orbit (closed traj.) in D



$$\underbrace{\int_{\partial S} f(x).n(x)\,dl}_{=0} = \iint_S \underbrace{\nabla.f(x)}_{\neq 0}\,dx$$

If $\nabla.f(x)$ does not equal zero or change sign the right hand side will never be zero. (# contradiction)

So there cannot be such a periodic orbit in D.

Ex

$\dot{x} = Ax$ $\quad$ $\text{div} f = \text{trace}(A)$

$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$ $\quad$ unless trace$(A) = 0 \Rightarrow$ no periodic orbits

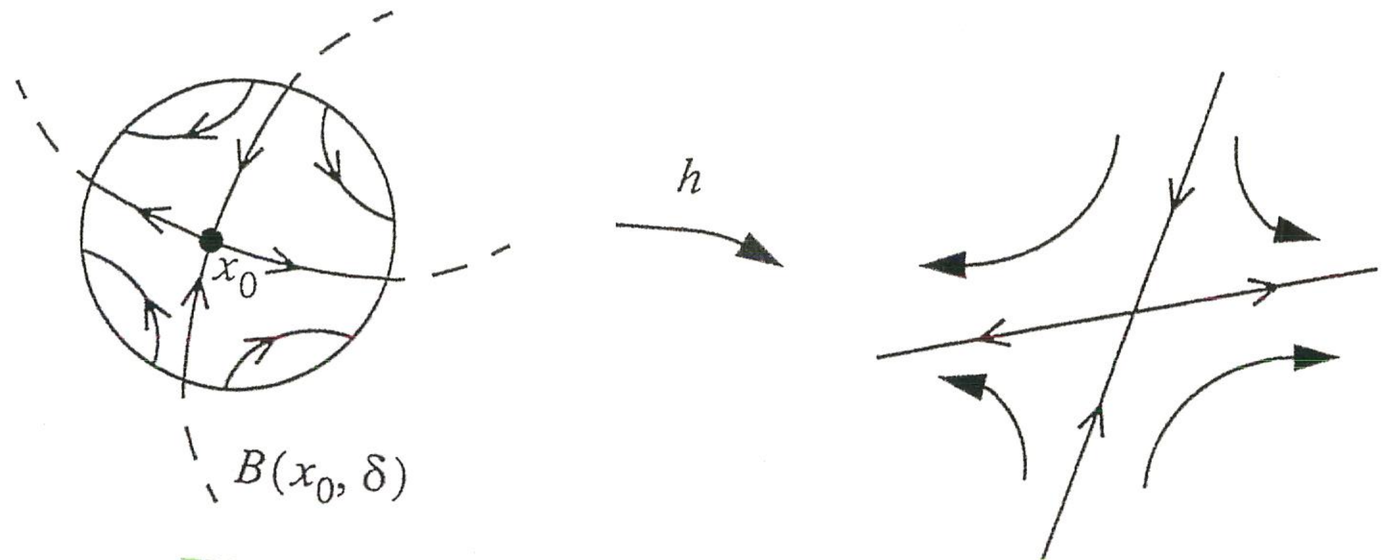


FIGURE 2.5. Illustrating the Hartman Grobman theorem