Nonlinear Systems

01/24/13

Lecture 2

Last time:

- Course mechanics
- Intro to nonlinear systems

Essentially nonlinear phenomena:

1) Finite escape time
2) Multiple isolated equilibria

Today:

3) limit cycles
4) Chaos

$\Big\}$ examples of

- Bifurcations (in 1st order systems)

Ex logistics equation

$$\dot{x} = \alpha\left(1 - \frac{x}{k}\right)x \quad ; \quad x(t) \in \mathbb{R} \quad \text{(1st order systems)}$$

Models population growth

$$(\alpha, K > 0)$$

Simpler model:

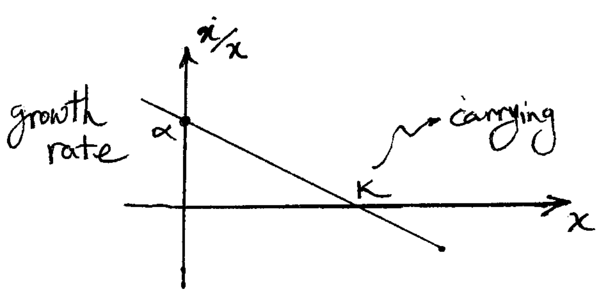
$\dot{x} = \alpha x$ $\quad$ α growth rate (positive number)

$x(t) = e^{\alpha t} x(0)$

$\dfrac{\dot{x}}{x} = \alpha$ $\quad$ (this model is based on assumption that the rate of change of population per capita is constant)

Issue: Population can grow unboundedly

Logistic equation provides a fix for this issue by assuming that $\dot{x}/x$ decays linearly w/ x (more sophisticated model would have more sophisticated decay functions)



Equilibrium points

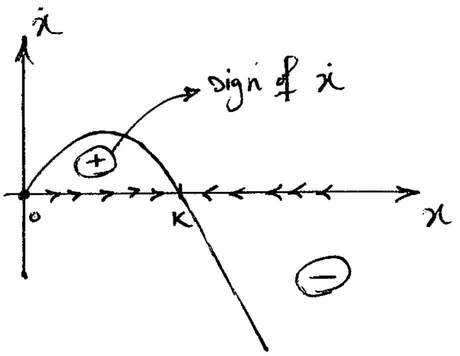
$$(\dot{\bar{x}} = 0) \Rightarrow f(\bar{x}) = \alpha\left(1 - \frac{\bar{x}}{K}\right)\bar{x} = 0 \Rightarrow \quad \bar{x} = 0 \text{ no population}$$

$$\bar{x} = K > 0 \text{ carrying capacity}$$

(yet another example of multiple isolated equilibria)

Linearization

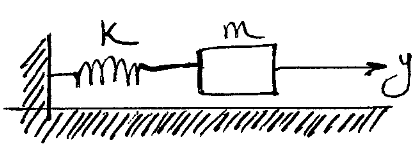
$$\frac{\partial f}{\partial x}\Big|_{\bar{x}} = \left(\alpha - \frac{2\alpha\bar{x}}{K}\right)\Big|_{\bar{x}} \quad \begin{cases} \xrightarrow{\bar{x}=0} \alpha \to \text{unstable} \\ \xrightarrow{\bar{x}=K} -\alpha \to \text{locally asymptotically stable} \end{cases}$$



no matter how big the population is, it will settle at the carrying capacity. However it will stay at 0 if it is at 0 initially (not self evolving)

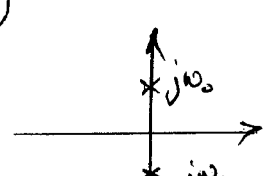
3) Limit cycles

Ex)

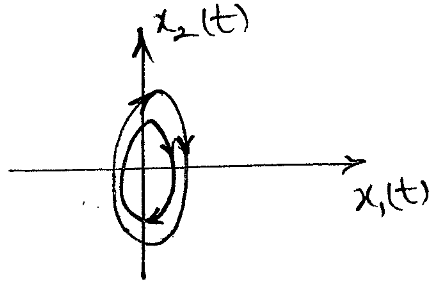


$m\ddot{y} + ky = 0$ (Harmonic oscillator), (LC circuit in EE)

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$\omega_0 = \sqrt{\frac{k}{m}}$

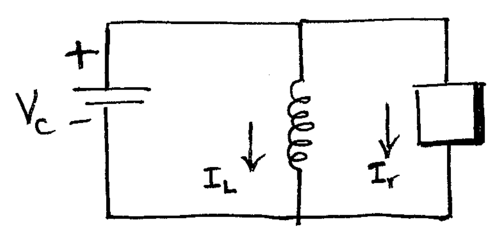


- Amplitude of oscillations depends on initial conditions.

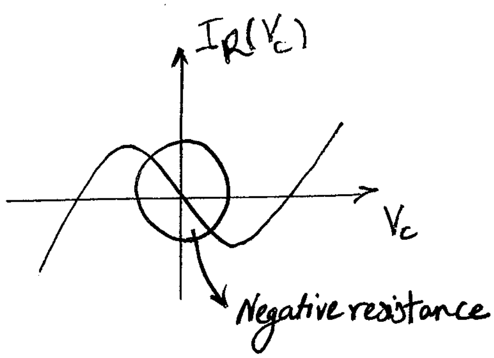
- Can be destroyed by small modeling imperfections (e.g. small amount of damping would decrease and bring oscillations to zero as $t \to \infty$.)

Moral: (structurally) robust oscillations are impossible in unforced LTI systems. (you need nonlinearity)

Ex Van der pol oscillator



$I_r = I_r(V_c)$



Ex $I_R(V_c) = -V_c + V_c^3$

$$\left.\begin{aligned} \dot{I}_L &= \frac{1}{L} V_c \\ \dot{V}_c &= -\frac{1}{C} I_L + \frac{1}{C}(V_c - V_c^3) \end{aligned}\right\} \text{Vanderpol oscillator}$$

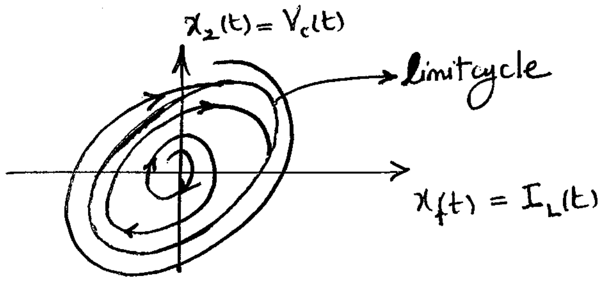
the origin is the unique e.p. here:

$$\begin{bmatrix} \bar{I}_L \\ \bar{V}_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearization around $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & +\frac{1}{C} \end{bmatrix}$$

* positive sign tells us that e.p. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an unstable e.p.



Structurally robust oscillator; no matter where you start you are going to be oscillating. (Apart from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$)

of course if you don't turn on the circuit you will not have oscillations.

4) Chaos (no precise definition)

Ex Lorenz system (attractor)

$$\left.\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= x(b-z)-y \\ \dot{z} &= xy-\tau z \end{aligned}\right\}$$ Simplified model of convective rolls in the atmosphere

The Lorenz system is a 3rd order system (3 states x, y, z)

a, b, τ : constant parameters

$a = 10$, $b = 28$, $c = 8/3$ (Chaos)

- no simple characterization of asymptotic behavior
- huge sensitivity to initial conditions

★ Bifurcations : "splitting into two branches"

translation (real meaning) : abrupt change (sudden) in qualitative behavior as parameters are varied.

creation (or death) of equilibrium points (or limit cycles) and/or change of their stability properties.

In the presence of parameters, even 1st order systems (i.e. scalar state) can have interesting properties.

3 types of bifurcations :

1. Fold (saddle-node ; blue sky)

$$\dot{x} = \alpha \pm x^2$$

$\hookrightarrow$ parameter $\alpha \in \mathbb{R}$

2. Transcritical

$$\dot{x} = \alpha x \mp x^2$$

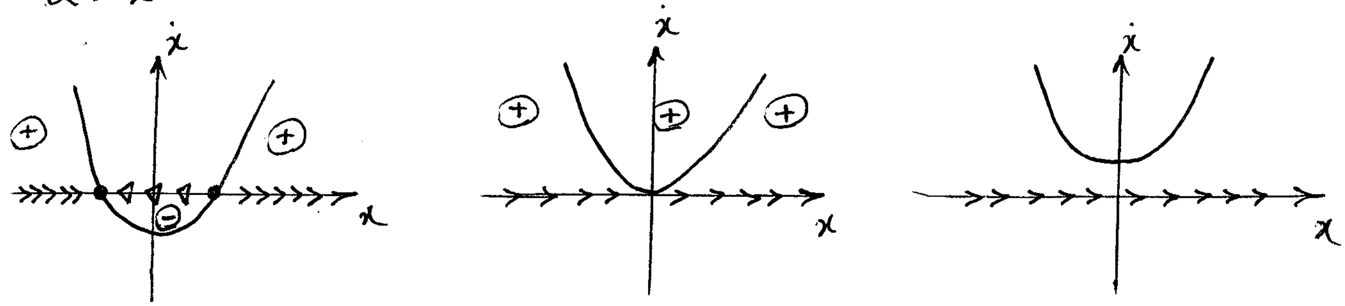
3. Pitch fork

$$\dot{x} = \alpha x \mp x^3$$

These can appear in higher order systems but essentially they are one dimensional.

1 Fold

$$\dot{x} = \alpha \pm x^2$$



a) $\alpha < 0$ b) $\alpha = 0$ c) $\alpha > 0$

Equilibrium points : $\bar{x} = \begin{cases} \pm\sqrt{|\alpha|} & \alpha < 0 \\ 0 & \alpha = 0 \\ \text{none} & \alpha > 0 \end{cases}$

Critical value of α: $\alpha_c = 0$

Linearization: $\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = 2\bar{x} = \pm 2\sqrt{|\alpha|}\,,\ \alpha < 0$

unstable

locally as. stable

Note! $A_c = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}_c = \bar{x}(\alpha_c)} = 0 \longrightarrow$ linearization disapears no info about stability of the system

(Also true for transcritical & pitch fork)

Bifurcation Diagram :