Nonlinear Systems

Lecture 2

Last time:

- Course mechanics
- Intro to nonlinear systems

Essentially nonlinear phenomena:

1) Finite escape time
2) Multiple isolated equilibria

Today:

3) Limit cycles
4) Chaos

Examples of:

- Bifurcations (in 1st order systems)

Ex - logistics equation

\[ \dot{x} = ax(1 - \frac{x}{K})x ; \quad x(t) \in \mathbb{R} \quad (1st \ order \ systems) \]

Models population growth

(\(a, K > 0\))
Simpler model:

\[ \dot{x} = \alpha x \]
\[ x(t) = e^{\alpha t} x(0) \]
\[ \frac{\dot{x}}{x} = \alpha \]

(this model is based on an assumption that the rate of change of population per capita is constant)

Issue: Population can grow unboundedly

Logistic equation provides a fix for this issue by assuming that \( \frac{\dot{x}}{x} \) decays linearly w/ \( x \) (more sophisticated model would have more sophisticated decay functions)

Equilibrium points

\[ (\dot{x} = 0) \Rightarrow f(x) = \alpha (1 - \frac{x}{K}) x = 0 \Rightarrow \bar{x} = 0 \text{ no population} \]
\[ \bar{x} = K > 0 \text{ carrying capacity} \]

(yet another example of multiple isolated equilibria)
Linearization

\[ \frac{\partial F}{\partial x} \bigg|_{x} = \left( \alpha - \frac{2x}{K} \right) \]

\[ x = K \rightarrow -\alpha \rightarrow \text{locally asymptotically stable} \]

\[ x = 0 \rightarrow \alpha \rightarrow \text{unstable} \]

\[ \text{no matter how big the population is, it will settle at the carrying capacity. However it will stay at 0 if it is at 0 initially (not self evolving).} \]

3) Limit cycles

Ex)

\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= -\frac{k}{m} y - \frac{1}{m} x
\end{align*} \]

\[ x_1 = y \]
\[ x_2 = \dot{y} \]

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \omega_0 = \sqrt{\frac{k}{m}} \]
Amplitude of oscillations depends on initial conditions.

Can be destroyed by small modeling imperfections (e.g., small amount of damping would decrease and bring oscillations to zero as $t \to \infty$.)

Moral: (structurally) robust oscillations are impossible in unforced LTI systems. (you need nonlinearity)

Ex. **Vander Pol oscillator**

\[
\begin{align*}
V_c & = -I_L \\
I_r & = I_r(V_c)
\end{align*}
\]

\[
\text{Negative resistance}
\]

\[
\begin{align*}
E_x & \quad I_R(V_c) = -V_c + V_c^3 \\
\hat{I}_L & = \frac{1}{L} V_c \\
\dot{V}_c & = -\frac{1}{C} I_L + \frac{1}{C} (V_c - V_c^3)
\end{align*}
\]

\[
\text{VanderPol oscillator}
\]
the origin is the unique e.p here:

\[
\begin{bmatrix}
I_L \\
V_c
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Linearization around (0)

\[
A = \begin{bmatrix}
0 & \frac{1}{C} \\
-\frac{1}{L} & \frac{1}{LC}
\end{bmatrix}
\]

* positive sign tells us that e.p. (0) is an unstable e.p.

\[x_L(t) = V_c(t)\]

\[x_f(t) = I_L(t)\]

Structurally robust oscillator, no matter where you start you are going to be oscillating.
(Apart from (0))

Of course if you don’t turn on the circuit you will not have oscillations.

4) Chaos (no precise definition)

Ex. Lorenz system (attractor)

\[
\begin{align*}
\dot{x} &= a(y-x) \\
\dot{y} &= x(b-z) - y \\
\dot{z} &= xy - \sigma z
\end{align*}
\]

Simplified model of convective rolls in the atmosphere
The Lorenz system is a 3rd order system (3 states $x, y, z$)

$a, b, \tau$: constant parameters

$a = 10$, $b = 28$, $c = \frac{8}{3}$ (Chaos)

- no simple characterization of asymptotic behavior
- huge sensitivity to initial conditions

* Bifurcations: "splitting into two branches"
  translation: abrupt change in qualitative behavior as parameters are varied.

  creation (or death) of equilibrium points (or limit cycles)
  and/or change of their stability properties.

In the presence of parameters, even 1st order systems (i.e. scalar state) can have interesting properties.
3 types of bifurcations:

1. Fold (saddle-node; blue sky)
   \[ \dot{x} = \alpha \pm x^2 \]
   \( \rightarrow \) parameter \( \alpha \in \mathbb{R} \)

2. Transcritical
   \[ \dot{x} = \alpha x \mp x^2 \]

3. Pitch Fork
   \[ \dot{x} = \alpha x \mp x^3 \]

These can appear in higher order systems but essentially they are one dimensional.

1. Fold
   \[ \dot{x} = \alpha \pm x^2 \]

   a) \( \alpha < 0 \)
   b) \( \alpha = 0 \)
   c) \( \alpha > 0 \)
Equilibrium points: \[
\bar{x} = \begin{cases} 
\pm \sqrt{|\alpha|} & \alpha < 0 \\
0 & \alpha = 0 \\
\text{none} & \alpha > 0
\end{cases}
\]

Critical value of $\alpha$: $\alpha_c = 0$  

Linearization: \[
\frac{d\bar{x}}{dx} \bigg|_{\bar{x}} = 2\bar{x} = \pm 2\sqrt{|\alpha|}, \quad \alpha < 0
\]

Note! \[
A_c = \frac{d\bar{x}}{dx} \bigg|_{\bar{x} = \bar{x}(\alpha_c)} = 0 \quad \rightarrow \quad \text{linearization disappears}
\]

(Also true for transcritical & pitchfork)

Bifurcation Diagram: