Nonlinear Systems

01/22/13

Lecture 1

Topics

			EE 5231
Analysis:	$\dot{x} = f(x)$	time invariant systems	$\dot{x} = Ax$
	$\dot{x} = f(x,t)$	time varing systems	$\dot{x} = A(t)x$
Control design:	$\dot{x} = f(x,u)$	time invariant	$\dot{x} = Ax + Bu$
	$\dot{x} = f(x,u,t)$	time varying	$\dot{x} = A(t)x + B(t)u$

Notation

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \; : \text{ state}$$

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix} \quad \text{input (control, disturbance)}$$

Example

1) $\dot{x} = \sin(x)$; $x(0) = x_0 \in \mathbb{R}$

$x(t) \in \mathbb{R}$ 1st order system (scalar state)

Equilibrium points:

(Points that system would stay in if left on its own)

solutions to $f(\bar{x}) = 0$

$(\bar{x} = \text{const} \Rightarrow \dot{\bar{x}} \equiv 0 = f(\bar{x})$

"points in the state space in which system will stay forever if it starts there."

$f(\bar{x}) = \sin(\bar{x}) = 0 \Rightarrow \bar{x} = k\pi \quad k = 0, \pm 1, \pm 2, \ldots$

Recall from linear systems

e.p. are solutions to $A\bar{x} = 0 \rightarrow \begin{cases} \bar{x} = 0 \\ \text{Null space of } A \end{cases}$

$A \in \mathbb{R}^{n \times n}$

if A not invertible

so for $A\bar{x} = 0$:

$\bar{x} = 0$ (unique e.p. if $\det(A) \neq 0$)

$\bar{x} = \mathrm{Null}(A)$: infinitely many e.p. if $\det(A) = 0$

We can use linearization around equilibrium points to study their <u>local</u> stability properties.

$$\dot{x} = f(x) \xrightarrow[\text{state} = \text{e.p.} + \text{fluctuations around it}]{x(t) = \bar{x} + \tilde{x}(t)} \overset{0}{\cancel{\dot{\bar{x}}}} + \dot{\tilde{x}} = f(\bar{x} + \tilde{x}) \overset{\text{Taylor series}}{=} \underset{0}{\cancel{f(\bar{x})}} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}}}_{A} \tilde{x} + \text{H.o.T.}$$

$$\Rightarrow \boxed{\dot{\tilde{x}} = A\tilde{x}}$$

a) $\mathrm{Re}(\lambda_i(A)) < 0, \ \forall i \Rightarrow \bar{x}$ is locally asymptotically stable

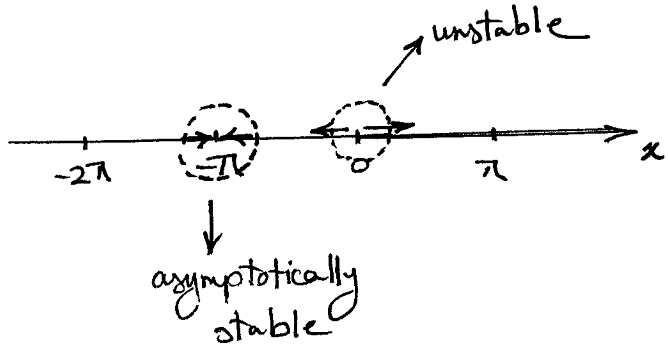
b) If there is i such that $\mathrm{Re}(\lambda_i(A)) > 0$

$\Rightarrow \bar{x}$ is unstable

Linearization of $\dot{x} = \sin(x)$

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \left.\cos(x)\right|_{\bar{x}} = \begin{cases} \cos(2K\pi) & = 1 \quad , n \text{ even} \\ \cos((2K+1)\pi) & = -1 \quad , n \text{ odd} \end{cases}$$

$$\bar{x} = n\pi \quad n = 0, \pm 1, \pm 2, \ldots$$

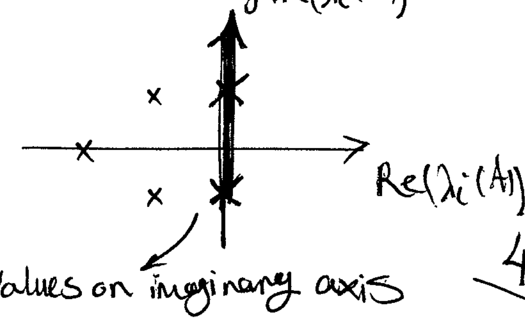


Issues

1) only provides local stability results
   (can't say anything about global properties)

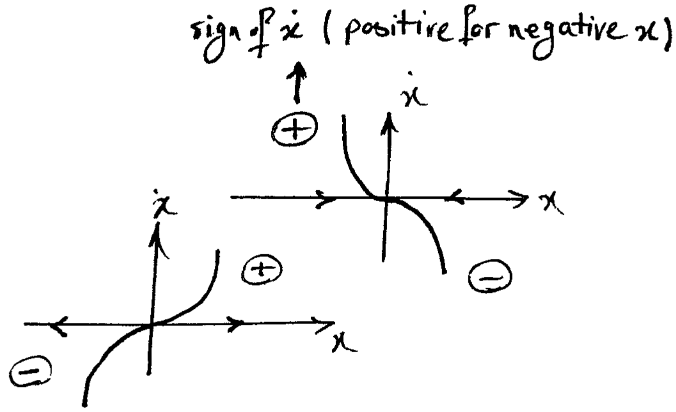
2) If $Re(\lambda_i(A)) \leq 0$ with some e-values having $Re(\lambda_i(A)) = 0$

linearization doesn't provide info about stability (need to examine H.O.T.)



Ex (a) $\dot{x} = -x^3$

(b) $\dot{x} = x^3$

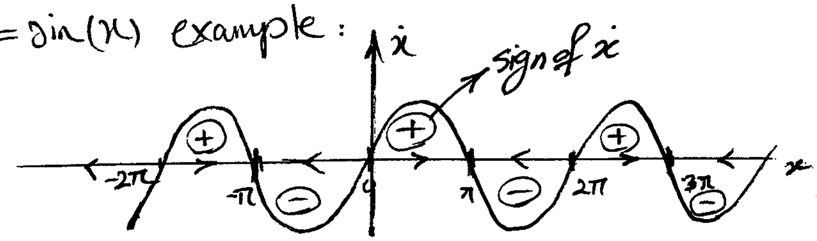


In both cases $A = 0$ $(\dot{\tilde{x}} = 0\tilde{x})$

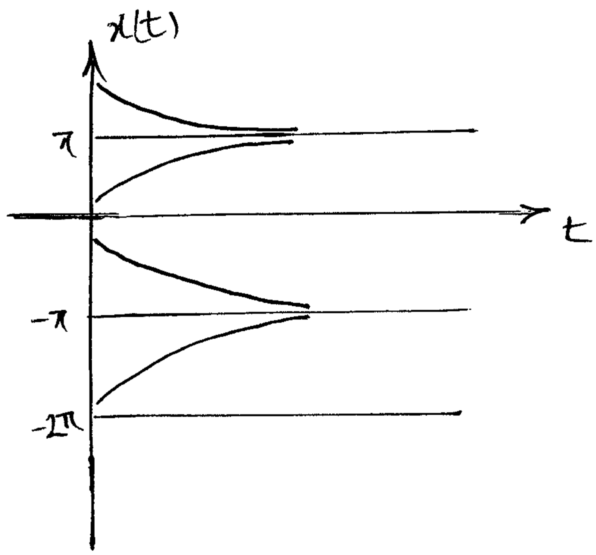
linearization does not provide any info about stability

$\begin{cases} \text{system (a): } \bar{x} = 0 \text{ is globally asymptotically stable} \\ \text{system (b): is unstable} \end{cases}$

→ In $\dot{x} = \sin(x)$ example:



Ex



"Essentially nonlinear phenomena"

1. Finite escape time

($x(t) \to \infty$, even for $t < +\infty$ finite t)

(Can't happen in linear case $x(t) = e^{At} x_0$)

Ex $\quad \dot{x} = x^2 \quad , x(t) \in \mathbb{R}$

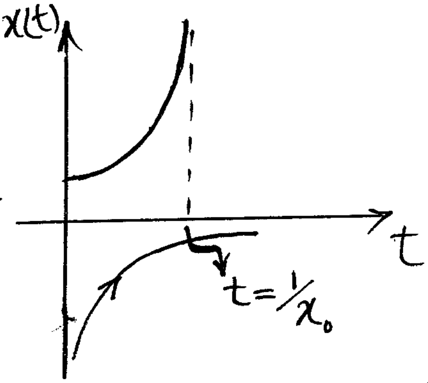
$$\frac{dx}{dt} = x^2 \Rightarrow \int_{x_0}^{x(t)} \frac{dx}{x^2} = \int_0^t dt$$

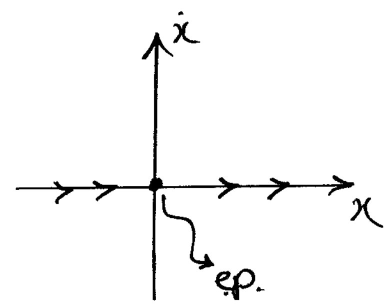
$$\Rightarrow -\frac{1}{x(t)} + \frac{1}{x_0} = t - 0$$

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

$$x_0 > 0 \Rightarrow t \to \frac{1}{x_0} \Rightarrow x(t) \to \infty$$

again linearization would not provide useful info here!



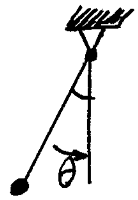


2. Multiple isolated equilibria

Ex

1) $\dot{x} = \sin(x)$

2) Pendulum



$$ml\ddot{\theta} + kl\dot{\theta} + mg\sin(\theta) = 0$$

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{k}{m}\dot{\theta}$$

$$\left.\begin{array}{l} x_1 = \theta \\ x_2 = \dot{\theta} \end{array}\right\} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{bmatrix}$$

e.p. $\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$; $k = 0, \pm 1, \pm 2, \dots$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(\bar{x}_1) & -\frac{k}{m} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -g/l & -k/m \end{bmatrix} & \text{stable, down} \\ \begin{bmatrix} 0 & 1 \\ +g/l & -k/m \end{bmatrix} & \text{unstable, up} \end{cases}$$