1. Consider the system:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(k_1 x_1 + k_2 x_2), \quad k_1, k_2 > 0,
\end{align*}
\]
where the nonlinearity \(g(\cdot)\) is such that
\[
g(y) y > 0, \quad \forall y \neq 0
\]
\[
\lim_{|y| \to \infty} \int_0^y g(\xi) \, d\xi = +\infty
\]
(a) Using an appropriate Lyapunov function, show that the equilibrium \(x = 0\) is globally asymptotically stable.

(b) Show that the saturation function \(\text{sat}(y) = \text{sign}(y) \min\{1, |y|\}\) satisfies the above assumptions for \(g(\cdot)\). What is the exact form of your Lyapunov function for this saturation nonlinearity?

(c) Parts (a) and (b) imply that a double integrator with a saturating actuator
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \text{sat}(u)
\end{align*}
\]
can be stabilized with the state-feedback controller \(u = -k_1 x_1 - k_2 x_2\). Design \(k_1\) and \(k_2\) to place the eigenvalues of the linearization at \(-1 \pm j\), and simulate the resulting closed-loop system both with, and without, saturation. Compare the resulting trajectories. (Please provide plots of \(x_1(t)\) and \(x_2(t)\) rather than phase portraits.)

2. Consider the mass-spring-damper system described by
\[
m\ddot{y} + \beta \dot{y} + ky = u,
\]
(a) If \(y(t)\) and \(u(t)\) are available for measurement, design a gradient algorithm to estimate constant but unknown parameters \(m\), \(\beta\), and \(k\).

(b) Simulate your algorithm in (a) assuming that true values are \(m = 20\), \(\beta = 0.1\), and \(k = 5\). Repeat your simulation for different choices of \(u(t)\) and observe the resulting parameter convergence properties.

3. Consider the reference model:
\[
\dot{y}_m = -ay_m + r(t), \quad a > 0,
\]
and the plant:
\[
\dot{y} = a^* y + b^* u, \quad b^* \neq 0.
\]
(a) Show that a controller of the form:
\[
u = \theta_1 y + \theta_2 r(t)
\]
with an appropriate choice of gains \(\theta_1^*\) and \(\theta_2^*\), drives the tracking error \(e := y - y_m\) asymptotically to zero.

(b) Now suppose \(a^*\) and \(b^*\) are unknown parameters, but the sign of \(b^*\) is known. Show that the adaptive implementation of the controller above achieves tracking when the gains are updated according to the rule:
\[
\begin{align*}
\dot{\theta}_1 &= -\text{sign}(b^*) \gamma_1 ye, \\
\dot{\theta}_2 &= -\text{sign}(b^*) \gamma_2 re,
\end{align*}
\]
where \(\gamma_1 > 0\) and \(\gamma_2 > 0\).
(c) Provide a condition that also guarantees \( \theta_1(t) \to \theta_1^* \) and \( \theta_2(t) \to \theta_2^* \) as \( t \to \infty \).

4. A simplified model of an axial compressor, used in jet engine control studies, is given by the following second order system
\[
\begin{align*}
\dot{\phi} &= -\frac{3}{2} \phi^2 - \frac{1}{2} \phi^3 - \psi \\
\dot{\psi} &= \frac{1}{\beta^2} (\phi + 1 - u).
\end{align*}
\]
This model captures the main surge instability between the mass flow and the pressure rise. Here, \( \phi \) and \( \psi \) are deviations of the mass flow and the pressure rise from their set points, the control input \( u \) is the flow through the throttle, and \( \beta \) is positive constant.

(a) Use backstopping to obtain a control law that stabilizes the origin \( (\phi, \psi) = 0 \).

(b) Use Sontag’s Formula and the Control Lyapunov Function obtained in part (a) to obtain an alternative control law.