Due Friday 04/19/13 (5pm, Xiaofan's office)

1. Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\,g(k_1 x_1 + k_2 x_2), \qquad k_1, k_2 > 0, \end{aligned}$$

where the nonlinearity $g(\cdot)$ is such that

$$g(y)\,y > 0, \quad \forall\, y \neq 0$$

$$\lim_{|y|\to\infty} \int_0^y g(\xi)\,\mathrm{d}\xi = +\infty$$

   (a) Using an appropriate Lyapunov function, show that the equilibrium $x = 0$ is globally asymptotically stable.

   (b) Show that the saturation function $\mathrm{sat}(y) = \mathrm{sign}(y)\min\{1, |y|\}$ satisfies the above assumptions for $g(\cdot)$. What is the exact form of your Lyapunov function for this saturation nonlinearity?

   (c) Parts $(a)$ and $(b)$ imply that a double integrator with a saturating actuator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mathrm{sat}(u) \end{aligned}$$

   can be stabilized with the state-feedback controller $u = -k_1 x_1 - k_2 x_2$. Design $k_1$ and $k_2$ to place the eigenvalues of the linearization at $-1 \pm j$, and simulate the resulting closed-loop system both with, and without, saturation. Compare the resulting trajectories. (Please provide plots of $x_1(t)$ and $x_2(t)$ rather than phase portraits.)

2. Consider the mass-spring-damper system described by

$$m\ddot{y} + \beta\dot{y} + ky = u,$$

   (a) If $y(t)$ and $u(t)$ are available for measurement, design a gradient algorithm to estimate constant but unknown parameters $m$, $\beta$, and $k$.

   (b) Simulate your algorithm in $(a)$ assuming that true values are $m = 20$, $\beta = 0.1$, and $k = 5$. Repeat your simulation for different choices of $u(t)$ and observe the resulting parameter convergence properties.

3. Consider the reference model:

$$\dot{y}_m = -a y_m + r(t), \qquad a > 0,$$

and the plant:

$$\dot{y} = a^\star y + b^\star u, \qquad b^\star \neq 0.$$

   (a) Show that a controller of the form:

$$u = \theta_1 y + \theta_2 r(t)$$

   with an appropriate choice of gains $\theta_1^\star$ and $\theta_2^\star$, drives the tracking error $e := y - y_m$ asymptotically to zero.

   (b) Now suppose $a^\star$ and $b^\star$ are unknown parameters, but the sign of $b^\star$ is known. Show that the adaptive implementation of the controller above achieves tracking when the gains are updated according to the rule:

$$\dot{\theta}_1 = -\mathrm{sign}(b^\star)\gamma_1 y e, \qquad \dot{\theta}_2 = -\mathrm{sign}(b^\star)\gamma_2 r e,$$

   where $\gamma_1 > 0$ and $\gamma_2 > 0$.

(c) Provide a condition that also guarantees $\theta_1(t) \to \theta_1^\star$ and $\theta_2(t) \to \theta_2^\star$ as $t \to \infty$.

4. A simplified model of an axial compressor, used in jet engine control studies, is given by the following second order system

$$\begin{aligned} \dot{\phi} &= -\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - \psi \\ \dot{\psi} &= \frac{1}{\beta^2}(\phi + 1 - u)\,. \end{aligned}$$

This model captures the main surge instability between the mass flow and the pressure rise. Here, $\phi$ and $\psi$ are deviations of the mass flow and the pressure rise from their set points, the control input $u$ is the flow through the throttle, and $\beta$ is positive constant.

(a) Use backstopping to obtain a control law that stabilizes the origin $(\phi, \psi) = 0$.

(b) Use Sontag's Formula and the Control Lyapunov Function obtained in part $(a)$ to obtain an alternative control law.