Due Th 01/31/13 (at the beginning of the class)

1. Duffing's equation,

$$\ddot{y} \,+\, \delta\,\dot{y} \,-\, y \,+\, y^3 \;=\; \alpha \cos(\omega t)\,,$$

exhibits chaotic behavior for certain values of parameters (δ, α, ω). For

$$\delta \;=\; 0.05, \quad \alpha \;=\; 0.4, \quad \omega \;=\; 1.3,$$

simulate this equation and

- Plot the resulting phase portrait (the graphical representation that illustrates the dependence of $x_2 := \dot{y}$ on $x_1 := y$ for various initial conditions);
- For your favorite choice of initial conditions, plot the time dependence of y and $\dot{y}$.

Discuss your observations.

2. Simulate the van der Pol equation

$$\ddot{y} \,+\, \left(y^2 \,-\, 1\right)\dot{y} \,+\, y \;=\; 0,$$

and

- Plot the resulting phase portrait;
- For your favorite choice of initial conditions, plot the time dependence of y.
- Compare your observations with the results obtained in the Duffing's equation problem.

Now, change the sign of the nonlinear term in the van der Pol equation

$$\ddot{y} \,-\, \left(y^2 \,-\, 1\right)\dot{y} \,+\, y \;=\; 0,$$

and determine local stability properties of the origin using linearization. For the same set of initial conditions that you chose in your simulations of the van der Pol equation, do simulations of this system. Discuss your observations.

3. Khalil, Problem 1.18 (attached).

4. Strogatz, Problems 3.4.2, 3.4.4, 3.4.7, and 3.4.9 (attached).

third equation is a torque equation for the shaft, with J as the rotor inertia and c_3 as a damping coefficient. The term $c_1 i_f \omega$ is the back e.m.f. induced in the armature circuit, and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux.

(a) For a separately excited DC motor, the voltages v_a and v_f are independent control inputs. Choose appropriate state variables and find the state equation.

(b) Specialize the state equation of part(a) to the field controlled DC motor, where v_f is the control input, while v_a is held constant.

(c) Specialize the state equation of part(a) to the armature controlled DC motor, where v_a is the control input, while v_f is held constant. Can you reduce the order of the model in this case?

(d) In a shunt wound DC motor, the field and armature windings are connected in parallel and an external resistance R_x is connected in series with the field winding to limit the field flux; that is, $v = v_a = v_f + R_x i_f$. With v as the control input, write down the state equation.

1.18 Figure 1.26 shows a schematic diagram of a magnetic suspension system, where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the, optically measured, ball position [211, pp. 192–200]. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers, and fast trains. The equation of motion of the ball is

$$m\ddot{y} = -k\dot{y} + mg + F(y, i)$$

where m is the mass of the ball, $y \geq 0$ is the vertical (downward) position of the ball measured from a reference point ($y = 0$ when the ball is next to the coil), k is a viscous friction coefficient, g is the acceleration due to gravity, $F(y, i)$ is the force generated by the electromagnet, and i is its electric current. The inductance of the electromagnet depends on the position of the ball and can be modeled as

$$L(y) = L_1 + \frac{L_0}{1 + y/a}$$

where L_1, L_0, and a are positive constants. This model represents the case that the inductance has its highest value when the ball is next to the coil and decreases to a constant value as the ball is removed to $y = \infty$. With $E(y, i) = \frac{1}{2}L(y)i^2$ as the energy stored in the electromagnet, the force $F(y, i)$ is given by

$$F(y, i) = \frac{\partial E}{\partial y} = -\frac{L_0 i^2}{2a(1 + y/a)^2}$$

When the electric circuit of the coil is driven by a voltage source with voltage v, Kirchhoff's voltage law gives the relationship $v = \dot{\phi} + Ri$, where R is the series resistance of the circuit and $\phi = L(y)i$ is the magnetic flux linkage.

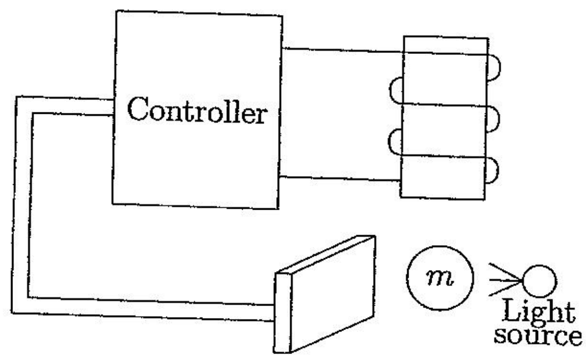

Figure 1.26: Magnetic suspension system of Exercise 1.18.

(a) Using $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = i$ as state variables and $u = v$ as control input, find the state equation.

(b) Suppose it is desired to balance the ball at a certain position $r > 0$. Find the steady-state values I_{ss} and V_{ss} of i and v, respectively, which are necessary to maintain such balance.

The next three exercises give examples of hydraulic systems [41].

1.19 Figure 1.27 shows a hydraulic system where liquid is stored in an open tank. The cross-sectional area of the tank, $A(h)$, is a function of h, the height of the liquid level above the bottom of the tank. The liquid volume v is given by $v = \int_0^h A(\lambda)\, d\lambda$. For a liquid of density ρ, the absolute pressure p is given by $p = \rho g h + p_a$, where p_a is the atmospheric pressure (assumed constant) and g is the acceleration due to gravity. The tank receives liquid at a flow rate w_i and loses liquid through a valve that obeys the flow-pressure relationship $w_o = k\sqrt{\Delta p}$. In the current case, $\Delta p = p - p_a$. Take $u = w_i$ to be the control input and $y = h$ to be the output.

(a) Using h as the state variable, determine the state model.

(b) Using $p - p_a$ as the state variable, determine the state model.

(c) Find u_{ss} that is needed to maintain the output at a constant value r.

1.20 The hydraulic system shown in Figure 1.28 consists of a constant speed centrifugal pump feeding a tank from which liquid flows through a pipe and a valve that obeys the relationship $w_o = k\sqrt{p - p_a}$. The pump characteristic for the specified pump speed is shown in Figure 1.29. Let us denote this relationship by $\Delta p = \phi(w_i)$ and denote its inverse, whenever defined, by $w_i = \phi^{-1}(\Delta p)$. For the current pump, $\Delta p = p - p_a$. The cross-sectional area of the tank is uniform; therefore, $v = Ah$ and $p = p_a + \rho g v / A$, where the variables are defined in the previous exercise.

In the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x^* vs. r.

3.4.1 $\dot{x} = rx + 4x^3$ **3.4.2** $\dot{x} = rx - \sinh x$

3.4.3 $\dot{x} = rx - 4x^3$ **3.4.4** $\dot{x} = x + \dfrac{rx}{1+x^2}$

The next exercises are designed to test your ability to distinguish among the various types of bifurcations—it's easy to confuse them! In each case, find the values of r at which bifurcations occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

3.4.5 $\dot{x} = r - 3x^2$

3.4.6 $\dot{x} = rx - \dfrac{x}{1+x}$

3.4.7 $\dot{x} = 5 - re^{-x^2}$

3.4.8 $\dot{x} = rx - \dfrac{x}{1+x^2}$

3.4.9 $\dot{x} = x + \tanh(rx)$

3.4.10 $\dot{x} = rx + \dfrac{x^3}{1+x^2}$