Lecture 24 12/06/13

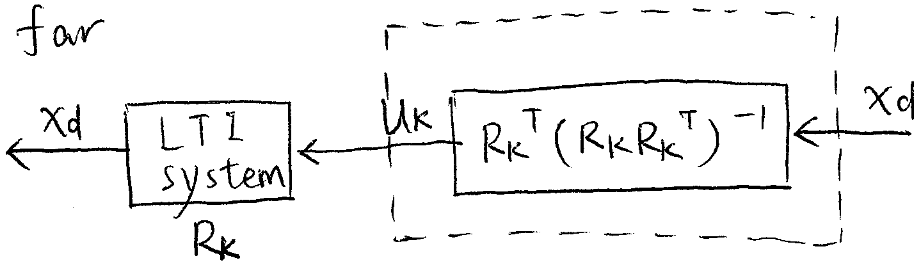
Last time:

- ctrl of cts time systems
- obsv (very briefly)

Today:

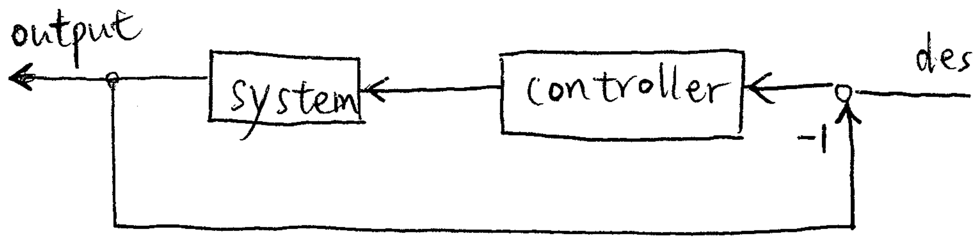
- "Pole placement" (state feedback design)
- Observer (estimator) design
- Output feedback design (Observer based design)

So far

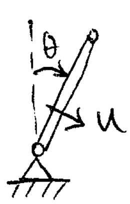


Problem: open-loop strategy
(doesn't pay attention to measurements)

Alternative: use feedback



State feedback:



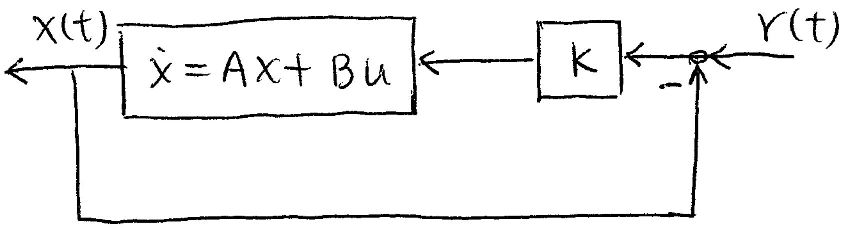
state $\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$\dot{x} = Ax + Bu$ ——— (1)

state feedback controller

$u(t) = -Kx(t)$



reference signal

$u(t) = -Kx(t) + r(t)$ ——— (2)

measured state

Control signal

state feedback matrix K

For $r(t) \equiv 0$, plug (2) $\rightarrow$ (1)

$\Rightarrow \dot{x}(t) = Ax(t) - BKx(t)$

$\Rightarrow \dot{x}(t) = \underbrace{(A-BK)}_{A_{cl}} x(t)$

Want to design K to move e-values ~~te~~ of A_{cl} to arbitrary location in the LHP.

Fact: This can be achieved iff system (1) is controllable

Matlab: place(A, B, Edes)

I.P. $u(t) = -Kx(t)$

$$= -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 1-k_1 & -k_2 \end{bmatrix}$$

$$f_{cl}(s) = \det(sI - A_{cl})$$
$$= \det \begin{bmatrix} s & -1 \\ k_1 - 1 & s+k_2 \end{bmatrix}$$
$$= s^2 + k_2 s + k_1 - 1$$
$$\overset{\text{want}}{=} (s+\lambda_1)(s+\lambda_2)$$
$$= s^2 + (\lambda_1 + \lambda_2) s + \lambda_1 \lambda_2$$
$$(= s^2 + 2\zeta w_0 s + w_0^2)$$

For closed-loop stability : $k_1 > 1$, $k_2 > 0$

Observer (Estimator) design

Last time

$$\dot{x} = Ax$$
$$y = Cx + w(t)$$
$$y(t) = Ce^{At} x_0$$

Rather than estimating the initial condition, we'll try to design a dynamical system (estimator/observer) where state should go to the "real state" as $t \to \infty$.

setup:

original system

$\dot{x} = Ax + Bu$ ---- (1) $\boxed{x(0) : \text{not known}}$

$u \hookrightarrow$ known input

$y = Cx$

Naive observer:

$\dot{\hat{x}} = A\hat{x} + Bu$ --- (2) $\boxed{\hat{x}(0) : \text{under my control}}$

$$\underbrace{\tilde{x}(t)}_{\text{observation error}} = x(t) - \underbrace{\hat{x}(t)}_{\text{observer state}}$$

(1) − (2)

$$\frac{d}{dt}(x - \hat{x}) = A(x - \hat{x}) \cancel{+ Bu - Bu}$$

$$\dot{\tilde{x}}(t) = A\tilde{x}(t)$$

$$\tilde{x}(0) \neq 0$$

Want $\hat{x}(t) \xrightarrow{t \to \infty} x(t)$

$$\Updownarrow$$

$$\tilde{x}(t) \xrightarrow{t \to \infty} 0$$

will hold iff $Re(\lambda_i(A)) < 0$, for all $i = 1, \cdots, n$

Fix of the naive observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{L(y(t) - \hat{y}(t))}_{\text{an "injection" term}}$$

$$\hat{y} = C\hat{x}(t)$$

$$\dot{\hat{x}} = A\hat{x} + Bu + LC(\underbrace{x(t) - \hat{x}(t)}_{\tilde{x}(t)}) \text{ --- (3)}$$

(1) − (3)

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) - LC\tilde{x}(t)$$

$$\boxed{\text{Error dynamics: } \dot{\tilde{x}}(t) = (A-LC)\tilde{x}(t)}$$

Fact: can design matrix L to move e-values of $A-LC$ to arbitrary location in LHP iff

(A, C) observable

$\Updownarrow$

pair (A^T, C^T) controllable

$\Updownarrow$

design $\bar{K}$, s.t. $\bar{A} - \bar{B}\cdot\bar{K}$ has desired e-values.

where $\bar{A} = A^T$, $\bar{B} = C^T$, $\bar{K} = L^T$.

$$\tilde{x}(t) = x(t) - \hat{x}(t) = e^{(A-LC)t}\tilde{x}(0)$$

I.P. Ex:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ 1-l_2 & 0 \end{bmatrix}$$

Fast convergence of $\hat{x}(t)$ to $x(t)$ requires "large" observer gain L.

Drawback : NOISE

Original system

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t) + \underbrace{w(t)}_{\to \text{ measurement noise}}$$

$$\dot{\tilde{x}}(t) = (A - LC)\,\tilde{x}(t) + L \cdot w(t)$$

fast convergence $\longrightarrow$ $\longleftarrow$ small noise amplification

Kalman filter : optimal observer

— Observer-based controller

(optimal feedback design)

state feedback

$$u(t) = -K\,x(t)$$

An option :

Use an observer-based controller of the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$u = -K\hat{x}$$

$$\dot{x} = Ax - BK\hat{x} \quad \text{------} \quad (a)$$

$$\dot{\hat{x}} = LCx + (A - BK - LC)\,\hat{x} \quad \text{---} \quad (b)$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

K, L design variables

Approach: rewrite

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

(a) $\Rightarrow \dot{x} = Ax - BK(x - \tilde{x})$

$= (A - BK)x + BK\tilde{x}$

(a) $-$ (b) $\Rightarrow \dot{\tilde{x}} = (A - LC)\tilde{x}$

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

If (A, B) controllable,
(A, C) observable,

can choose K & L s.t. e-values of A_{cl} @ arbitrary location