12/3 Lecture 23

Ages ago:

- reachability for LTI DT systems $\quad x_d \xleftarrow{} \boxed{R_K} \xleftarrow{u_K} \boxed{R_K^T (R_K R_K^T)^{-1}} \xleftarrow{} x_d$
- kalman rank test
- rank $\underbrace{[A^{n-1}B \mid \dots \mid B]}_{R_n} = n$ (# of states)
- PBH test
- rank $[zI - A \mid B] = n$ for all e-values of A
  $z = \lambda_1(A), \dots, \lambda_n(A)$
- reachability gramian
  $P_K = R_K R_K^T = \sum_{i=0}^{K-1} A^i B B^T (A^i)^T$
  ↳ state transition matrix

  $P_{K+1} = A P_K A^T + B B^T$
- minimum energy problem
  $\min u_K^T u_K$ s.t. $x_d - R_K u_K = 0$

controllability: go from $x(0) \neq$ ⊚
to $x(k) = 0$

$x(k) = A^k x(0) + R_K u_K$, with $x(k) = 0$

$\Rightarrow -A^k x(0) = R_K u_K$

no difference between reach / control if A invertible

Reachability in Continuous Time

$\dot{x}(t) = A x(t) + B u(t)$

$x(T) = e^{AT} x(0) + \int_0^T e^{A(T-t)} B u(t) \, dt$, with $x(0) = 0$

can we choose $u[0, T]$ s.t. $x(T) = x_d = [R_T u](T)$

$\begin{pmatrix} \# \\ \# \\ \# \end{pmatrix} \quad x_d \xleftarrow{} \boxed{R_T} \xleftarrow{} u$

→ maps infinite funct. to finite vector

$P_T = R_T R_T^{?}$

$= \int_0^T e^{At} B B^T e^{A^T t} \, dt$

$\frac{dP_T}{dT} = A P_T + P_T A^T + B B^T$

↳ differential Lyapunov eq

If A is Hurwitz $\Rightarrow A P_\infty + P_\infty A^T = B B^T$

$\hookrightarrow$ algebraic Lyaponov eq.

$$\boxed{P_\tau = R_\tau R_\tau^{ad}}$$

$$x_d \longleftarrow \boxed{R_\tau} \xleftarrow{\;u\;} \boxed{R_\tau^{ad}(R_\tau R_\tau^{ad})^{-1}} \longleftarrow x_d$$

$R_\tau^{ad} = \cancel{B} B^T e^{A^T t}$

$R_\tau : L_2[0, \tau] \rightarrow \mathbb{R}^n$

$R_\tau^{ad} \cdot z = B^T e^{A^T t} \cdot z$

minimum energy control:

$$\boxed{u_{opt}(t) = B^T \cdot e^{A^T t} \cdot P_\tau^{-1} \cdot x_d}$$

$x_d = [R_\tau u](\tau)$

$$\begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} * & & \\ * & & \\ * & \underline{\text{operator}} & \\ * & & \\ * & \longrightarrow & \infty \end{bmatrix} \begin{bmatrix} \downarrow \\ \infty \end{bmatrix}$$

no size of R_τ

$\Rightarrow$ it is an operator

$u(0, \tau)$

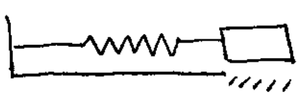
minimize $\int_0^\tau u^T(t) u(t) \, dt \quad$ s.t. $x_d - \int_0^\tau e^{A(\tau - t)} B u(\tau) \, d\tau = 0$

$\boxed{\text{Observability}}$

DT: $x(k+1) = A x(k) \quad \ldots (1)$

$y(k) = C \cdot x(k) \ldots (2)$ (measured output)

ex//



$$\begin{bmatrix} p(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} p(k) \\ u(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p(k) \\ u(k \end{bmatrix}$$

$x(k) = A^k \cdot x_0 \quad \ldots (3)$

$(3) \rightarrow (2)$: $y(k) = C A^k \cdot x_0$

$k = 0 \Rightarrow y(0) = C \cdot x_0$

$k = 1 \Rightarrow y(1) = C \cdot A \cdot x_0$

$\vdots$

$k = j \Rightarrow y(j) = C A^j x_0$

$$\Rightarrow \begin{bmatrix} y(0) \\ \cdot \\ \cdot \\ \cdot \\ y(k-1) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{k-1} \end{bmatrix}}_{\hookrightarrow O_k} x_d$$

ex// continued...

$\text{Null}(O_k)$ = unobservable subspace

System: $x(k+1) = A\,x(k)$
$\qquad\quad y(k) = C\,x(k)$

observable $\Leftrightarrow$ rank $O_n = n$

$\Updownarrow$

rank $O_n^T = n$

$$O_n^T = \left[\, C^T \; A^T C^T \;\vdots\; \cdots \;\vdots\; (A^T)^{n-1} C^T \,\right]$$

$\Downarrow$

$z(k+1) = A^T z(k) + C^T u(k)$
is reachable