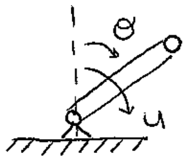
Lecture 20 11/19

Today: Reachability of LTI DT systems
↳ "controlability"
↳ stear initial condition $x(0) \neq 0$ to 0 in k steps

LTI DT:

$$x(k+1) = A x(k) + B u(k) \quad \dots \quad (1)$$



$$x := \begin{bmatrix} \theta \\ \ddot{\theta} \end{bmatrix}$$

Solution to (1) given by:

$$x(k) = A^k x(0) + \sum_{i=0}^{k-k} A^{k-i-1} B \cdot u(i)$$

Q: given $x(0) = 0$ and desired state $x_d = x$ desired, can we choose a sequence of inputs $\{u(0), \dots u(k_f - 1)\}$ to bring state @ time k_f into x_d

$$\underbrace{x(k_f)}_{\text{given}} = \underbrace{x_d}_{\text{given}} = \sum_{i=0}^{k_f-1} A^{k_f-i-1} B \, u(i)$$

(if we have m states + n inputs: $A \in \mathbb{R}^{m \times m}$ $B \in \mathbb{R}^{m \times n}$)

$$= \underbrace{\left[A^{k_f-1} B \;\vdots\; A^{k_f-2} B \;\vdots\; \dots \;\vdots\; AB \;\vdots\; B \right]}_{R_{k_f} \to \text{not square, not invertible}} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(k_f-1) \end{bmatrix}_{n \cdot k_f} \longrightarrow u_{k_f}$$

$$\underbrace{x_d}_{\text{given}} = \underbrace{R_{k_f}}_{\text{given}} \cdot u_{k_f} = R_k \cdot \underbrace{u_k}_{\text{variable}} \quad \dots (*)$$

We can choose u_k to satisfy $(*)$ iff $x_d \in \text{Range}(R_k)$

recall: $\text{Range}(A) = \{ y \text{ s.t. } y = A \cdot x \text{ for all } x \}$

Need to study Properties of matrix R_k

Some observations:

$$x(0) = 0 \Rightarrow x(k+1) = R_{k+1} \cdot u_{k+1} = \underbrace{\left[A^k \cdot B \;\vdots\; R_k \right]}_{R_{k+1}} \cdot u_k$$

↳ $\text{Range}(R_k) \subseteq \text{Range}(R_{k+1})$

more generally, $\text{Range}(R_k) \subseteq \text{Range}(R_\ell) \quad \forall \ell \geq k$

Q: Is there value K after which $\text{range}(R_\ell)$ w/ $\ell > k$ will saturate? Yes!

Cayley-Hamilton Thm:

Any $A \in \mathbb{R}^{n \times n}$ (square) satisfies its own characteristic equation: $f(s) = \det(sI - A)$

$$= s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

$$= 0$$

$$\Rightarrow A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I = 0$$

Note: If $k = n$ where $x(t) \in \mathbb{R}^{\boxed{n \rightarrow \# \text{ of states}}}$

$$\Rightarrow R_{n+1} = [A^n \cdot B \mid R_n]$$

$$[A^{n-1} B \mid \dots AB \mid B]$$

$$\text{Range}(R_k) \leq \text{Range}(R_n) = \text{Range}(R_\ell)$$

$$k \leq n \leq \ell$$

def: A system $x(k+1) = Ax(k) + Bu(k)$ is reachable if any $x_d \in \mathbb{R}^n$ is reachable in n or fewer states.

Thm (Kalman Rank Test)

$x(k+1) = Ax(k) + Bu(k)$ reachable $\Leftrightarrow$

$\text{Rank}(R_n) = n$

For single input systems $B \in \mathbb{R}^{n \times 1}$

$A^k B \in \mathbb{R}^n$

$R_n = [A^{n-1} B \dots B] \in \mathbb{R}^{n \times n}$

$\boxed{\det R_n \neq 0 \Leftrightarrow \text{rank}(R_n) = n}$

ex//

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \overset{A}{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \overset{B}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}} \cdot u(k)$$

$n = 2$

$$R_n = R_2 = [A^{2-1} B \mid A^{2-2} B] = [AB \mid B]$$

$$A \cdot B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 b_1 \\ \lambda_2 b_2 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \lambda_1 b_1 & b_1 \\ \lambda_2 b_2 & b_2 \end{bmatrix} \Rightarrow \det(R_2) = \lambda_1 b_1 b_2 - \lambda_2 b_1 b_2 = (\lambda_1 - \lambda_2) b_1 b_2$$

$\boxed{\text{Not Reachable}} \Leftrightarrow \lambda_1 = \lambda_2$ or $b_1 = 0$ or $b_2 = 0$

ex// Continued

a.) $b_2 = 0$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \cdot u(k)$$

$$x_2(k+1) = \lambda_2 \cdot x_2(k) \Rightarrow x_2(k) = \lambda_2^{k} \, x_2(0)$$

x_2 cannot be controlled!

b.) $\lambda_1 = \lambda_2 = \lambda$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cdot u(k)$$

$$R_2 = \left[\begin{array}{c|c} \lambda b_1 & b_1 \\ \lambda b_2 & b_2 \end{array}\right] \Rightarrow \text{Range}(R_2) = \alpha \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} ;\ \alpha \in \mathbb{R}$$

$$= [\lambda \cdot B \,\vdots\, B]$$