Lecture 19 11/12

Today: Forced Response Slideshow

freq. response tells you how amp change

maps components of $\hat{d}(\omega) \rightarrow \hat{y}(\omega)$

forier trasform of input " " output

Singular Value decomposition:

$$\underset{m \times n}{\begin{bmatrix} H(\omega) \end{bmatrix}} = \underset{m \times m}{\begin{bmatrix} u_1, \dots u_p \end{bmatrix}} \underset{n \times}{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}} \begin{bmatrix} v_1^* \\ \vdots \\ v_m^* \end{bmatrix}$$

doesn't have to go to bottom

unitary matrix → columns mutually orthogonal

same goes for $v \Rightarrow v^{-1} = v^*$ + $u^{-1} = u^*$

$u_i(\omega)$ - left singular vectors

$v_i(\omega)$ - right singular vectors

$$H(\omega) H^*(\omega) u_i(\omega) = \sigma_i^2(\omega) u_i(\omega)$$

$$H^*(\omega) H(\omega) v_i(\omega) = \sigma_i^2(\omega) v_i(\omega)$$

$$\hat{y}(\omega) = H(\omega)\hat{d}(\omega) = \sum_{i=1}^{n} \sigma_i(\omega) u_i(\omega) \langle v_i(\omega), \hat{d}(\omega) \rangle$$

NOT IN SLIDESHOW

a looong time ago: $\dot{x}(t) = Ax(t)$ A: diagonalizable

$$x(t) = e^{At} \cdot x_0(t)$$

$$= V \cdot e^{\Lambda t} \cdot W^* \cdot x_0$$

$$= \sum_{i=1}^{n} v_i e^{\lambda_i t} \underbrace{w_i^* x_0}_{\text{scalar} = \alpha_i}$$

$$\Rightarrow x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} \begin{bmatrix} v_i \end{bmatrix}$$

scalar

scalar denotes advance in time

today $\to$ $\hat{y}(\omega) = H(\omega)\,\hat{d}(\omega)$

$\hookrightarrow \dot{x} = Ax + by$

$y = Cx$

$$\hat{y}(\omega) = U \Sigma V^* \cdot \hat{d}(\omega)$$
$$= \sum_{i=1}^{r} u_i \cdot \sigma_i \underbrace{v_i^* \hat{d}(\omega)}_{\text{scalar } B_i} = \sum_{i=1}^{r} \sigma_i B_i \begin{bmatrix} u_i \end{bmatrix}$$
$$= \sum_{i=1}^{r} B_i(\omega)\,\sigma_i(\omega) \underbrace{\begin{bmatrix} u_i(\omega) \end{bmatrix}}_{\text{orthogonal}}$$

note: if $\hat{d}(\omega) := v_j \Rightarrow \hat{y}(\omega) = \sum_{i=1}^{r} u_i \sigma_i \underbrace{v_i^* v_j}_{\delta_{ij}} = \sigma_j(\omega)\, u_j(\omega)$

since $\sigma_1(\omega) \geq \sigma_2(\omega) \geq \sigma_3(\omega) \geq \ldots \geq \sigma_r(\omega) > 0$

$\Rightarrow$ the largest amplification @ any ω is given by σ_1

$(\sigma_1(\omega) = \sigma_{max}(H(j\omega))$

Note: If $\hat{d}(\omega) = v_j \Rightarrow \|\hat{d}(\omega)\|_2^2 = v_j^* v_j = 1$

$\|\hat{y}(\omega)\|_2^2 = \hat{y}(\omega)^* \cdot \hat{y}(\omega) = \sigma_j^2(\omega)$

"(2-induced norm)"

back to slideshow...

jk! back to NOT IN SLIDESHOW

note- since v_j's form orthogonal basis of the output space, we can write:

$$\hat{d}(\omega) = \sum_{j=1}^{m} a_j(\omega) \cdot v_j(\omega)$$

how to determine singular values:

$H = U \cdot \Sigma \cdot V^*$

$H \cdot H^* = U \Sigma V^* V \Sigma^* U^* = U \underbrace{\Sigma \Sigma^*}_{\Lambda} U^*$

$H \cdot H^* \cdot U = U \cdot \Sigma \Sigma^*$; $U = [u_1, \ldots u_p]$

$H \cdot H^* u_i = \sigma_i^2 \cdot u_i$

Matlab $\to$ e-value d-comp of $H, H^* \to [U, \Lambda]$

back to slideshow ...

$L_2 \rightarrow$ energy gain

"pseudospectra" of linear systems

$\dot{x}(t) = (A + B\Gamma C)\, x(t)$

↳ modeling uncertainty

large amp
worst case $\Rightarrow$ small stability margins

Toy example:

"Frobrinius Norm" (spelling)

↳ of Matrix → trace (HH^*)

sum of squares of singular values