Make-up Lecture 11/8

Today: $\xrightarrow{u} \boxed{\text{LTI system}} \xrightarrow{y}$

Consider finite energy input systems

$\Rightarrow \|u\|_2^2 = \int_0^\infty \|u(t)\|_2^2\, dt = \int_0^\infty u^T(t)\cdot u(t)\,dt = \int_0^\infty \sum_{i=1}^{m} |u_i(t)|^2 dt < \infty$

output :

$$y(t) = \int_0^t \underbrace{H(t-\tau)}_{\text{impulse response}}\cdot u(\tau)\,d\tau$$

$\downarrow$ fourier transform

$Y(j\omega) = H(j\omega)\cdot U(j\omega)$

↳ study energy of output: $\|y\|_2^2 = \int_0^\infty y^T(t)\cdot y(t)\,dt$

and remember → transforms preserve norms

$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(j\omega)\,Y(j\omega)\,d\omega$

↳ Parsevals equality

ex// $Y = \begin{bmatrix} 1+3j \\ 2-4j \end{bmatrix} \Rightarrow Y^* = (\bar{Y})^T = \begin{bmatrix} 1-3j & 2+4j \end{bmatrix}$

$$\Rightarrow \|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^*(j\omega)\,H^*(j\omega)\,H(j\omega)\,U(j\omega)\,d\omega$$

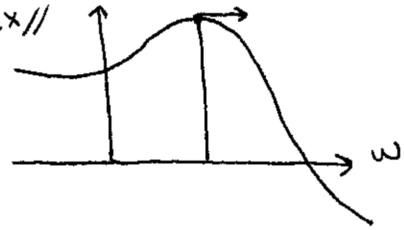
In SISO case, $U(j\omega)$, $Y(j\omega)$, $H(j\omega) \in \mathbb{C}$ at each ω

$$\Rightarrow \|y\|_2^2 = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} |H(j\omega)|^2\cdot|U(j\omega)|^2 \le \sup_\omega |H(j\omega)|^2\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}|U(j\omega)|^2 d\omega$$
$$\le \sup_\omega |H(j\omega)|^2\cdot\|u\|_2^2$$

If $\|u\|_2^2 \ne 0 \Rightarrow \|y\|_2^2 / \|u\|_2^2$ = output energy/input energy

$$\le \underbrace{\sup_\omega |H(j\omega)|^2}_{\text{peak on Bode mag. plot}}$$

ex//



$H(s) = \frac{k}{s+1} \Rightarrow$

$\sup_{\omega} |H(j\omega)|^2 = k^2$

Want to determine "worst case" ratio of $\|y\|_2^2 / \|u\|_2^2$

$\sup \|y\|_2^2 / \|u\|_2^2 \quad ; \quad 0 < \|u\|_2^2 \leq 1$

[street lingo: "maximize output/input (energy) under constraint that $0 \neq$ input energy ≤ 1

(turns out) $= \sup_{\omega} |H(j\omega)|^2$

BIG Q: What do we do in MIMO case?

(what is equivalent of $\sup_{\omega} |H(j\omega)|^2$ in MIMO system?)

$\Rightarrow \boxed{\sup_{\omega} \sigma^2_{max}(H(j\omega))}$

↳ largest singular value of $H(j\omega)$

BIG ASIDE: Singular Value Decomposition (of matrix M)

but only $M =$ square

→ for square matrix we can do e-value decomposition

$M \in \mathbb{C}^{n \times n} \qquad M \cdot v_i = \lambda_i v_i$

→ If M is Hermitian $(M = M^*) \Rightarrow V^{-1} = V^*$ (unitary diag.)

$\Rightarrow M = V \cdot \Lambda \cdot V^{-1} \equiv V \Lambda V^*$

Q: how do we figure out "personality" of rectangular matrix?

(e-value decomp reveals personality of square matrix)

↳ Fact of Life (oh + Linear Algebra)

ANY matrix $M \in \mathbb{C}^{m \times n}$ can be represented by:

$$m\left\{\left[\underbrace{\quad M \quad}_{n}\right]\right. = m\left\{\left[\underbrace{U}_{m}\right]\right. \left[\underbrace{\quad \Sigma \quad}_{n}\right] \left[\quad V^* \quad\right]\Big\} n$$

where $UU^* = U^*U = I_m$

$VV^* = V^*V = I_n$

Σ - matrix of singular values