Lecture 12 10/15

today: stability of LTI systems

any square matrix can be brought into $J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m \end{bmatrix}$

⇒ block diagonal form

Where J_i's are either:

- Jordan block ex// $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$; $\lambda \in \mathbb{C}$
- diagonal Matrix ex// $\begin{bmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$; $\lambda_i \in \mathbb{C}$

remaining task: choice of coordinate transformation that brings A to J.

$A = T \cdot J \cdot T^{-1}$

$T^{-1} \cdot A \cdot T = J$

NOTE: If there are no Jordan blocks in J

(i.e. if $J = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$)

choose $T = V = [v_1, \dots v_n]$: matrix of e-vectors

$A \cdot v_i = \lambda_i v_i$

complete aside → dilemma: $\begin{bmatrix} \sigma_i + j\omega_i & 0 \\ 0 & \sigma_i - j\omega_i \end{bmatrix}$ or $\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}$

doesn't matter which you use!

Challenge: how to choose T when A doesn't have a full set of linearly independent e-vectors.

(ie. $\det [v_1 \dots v_n] = 0$)

then we can't choose $T = V$

go back to $A \cdot T = T \cdot J$

→ rewrite as: $A \cdot [T_1, T_2, \ldots T_m] = [T_1, T_2, \ldots T_m] \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$

⇒ $A \cdot T_i = T_i \cdot J_i$; $i = 1, 2, \ldots, m$

NOTE: If $J_i = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}$

$T_i = [v_1, \ldots, v_r]$

$A \cdot v_j = \lambda_j v_j$

If J_i is a Jordan Block ex// $J_i = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$A [v_{i1} \; v_{i2} \; v_{i3}] = [v_{i1} \; v_{i2} \; v_{i3}] \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$[\underline{A v_{i1}} \mid A v_{i2} \mid A v_{i3}] = [\underline{\lambda \cdot v_{i1}} \mid v_{i1} + \lambda v_{i2} \mid v_{i2} + \lambda v_{i3}]$

⇒ $A v_{i1} = \lambda v_{i1}$

$A v_{ij} = \lambda v_{ij} + v_{i(j-1)}$; $j = 2, 3$

↳ generalized e-vectors

Modal Conditions for Stability of LTI Systems

$x^+(t) = At$ ... (1)

↳ $x^+(t) = \begin{cases} \frac{dx(t)}{dt} & \text{in CT} \\ x(t+1) & \text{in DT} \end{cases}$

Introduce change of coordinates to bring (1) to:

$z^+(t) = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix} \cdot z(t)$

for now, assume that $J = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$

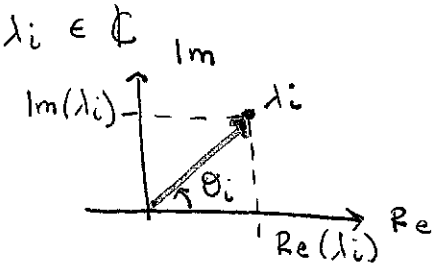
→ (back to full generality in a moment)

for diagonalizable A:

$$z_i(t) = \begin{cases} e^{\lambda_i t} \cdot z_i(0), & CT \\ \lambda_i^{t} \cdot z_i(0), & DT \end{cases}$$

$$z(t) := \begin{bmatrix} z_i(t) \\ \vdots \\ z_n(t) \end{bmatrix}; \quad z_i(t) \rightarrow \text{scalar}$$

$\lambda_i \in \mathbb{C}$



$$\lambda_i = \underbrace{Re(\lambda_i) + Im(\lambda_i)}_{\rightarrow \text{continuous}} = \underbrace{|\lambda_i| e^{j\arg(\lambda_i)} = |\lambda_i| e^{j\theta_i}}_{\hookrightarrow \text{discrete}}$$

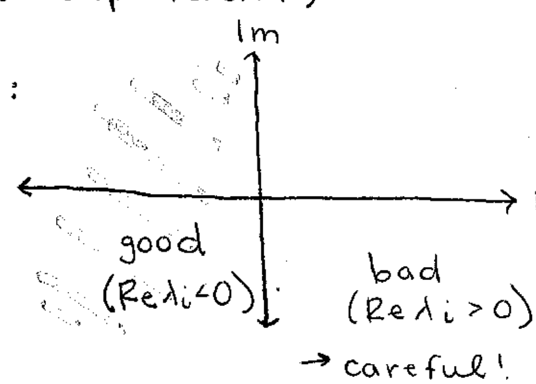
CT: $z_i(t) = e^{\lambda_i t} \cdot z_i(0) = e^{(Re(\lambda_i) + j\, Im(\lambda_i))t} \cdot z_i(0)$
$= e^{Re(\lambda_i t)} \cdot e^{j\, Im(\lambda_i) t} \cdot z_i(0)$

DT: $z_i(t) = \lambda_i^t \cdot z_i(0) =$
$= |\lambda_i|^t \, e^{j\arg(\lambda_i) \cdot t} \cdot z_i(0)$

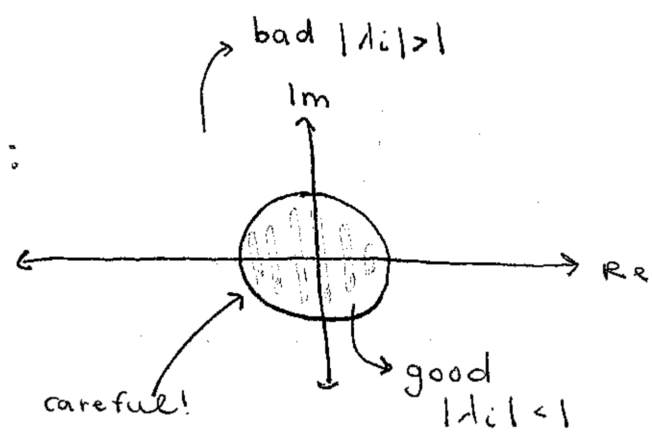
→ determines whether decay or growth

Q: for all $z_i(0)$, determine conditions for growth / decay of $|z_i(t)|$.
(λ_i - dependent)

CT:



DT:



note: If we have $J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \underset{\text{in CT}}{\Rightarrow} e^{Jt} = e^{\lambda t} \cdot \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$

Same conclusion for $Re(\lambda) > 0$ & $Re(\lambda) < 0$
growth (for $Re(\lambda) > 0$); ↳ decay (goes to 0 as $t \to \infty$)
$Re(\lambda) = 0$

ex// $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 2 e-values both sit @ zero

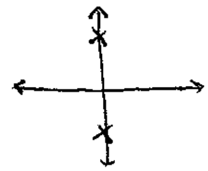
$\Rightarrow e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$(\lambda_1 = \lambda_2 = 0)$

algebraic multiplicity 2

$\Rightarrow$ growth in time (goes to ∞ as $t \to \infty$)

ex// $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda_1 = j$, $\lambda_2 = -j$



$\Rightarrow$ oscillates forever

ex// $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda_1 = -1$, $\boxed{\lambda_2 = +1}$

$\Rightarrow$ growth in time

CT case: $\dot{x}(t) = Ax$

1. Stable: $Re(\lambda_i) < 0$ for all i
   - $\rightarrow$ all trajectories are bounded + decay to zero as $t \to \infty$
2. Unstable:
   1.) there is i such that $Re(\lambda_i) > 0$
   2.) there is i such that $Re(\lambda_i) = 0$ + alg. mult. > 1