10/8 Lecture 10

Modal Form of LTI system: (slide show)

State Space Eq: $\dot{x}(t) = A\,x(t) + B\,d(t)$
$y(t) = C\,x(t)$

Solution:
$x(t) = \underbrace{e^{At}\,x(0)}_{\text{unforced}} + \underbrace{\int_0^t e^{A(t-\tau)}\,d(\tau)\,d\tau}_{\text{forced}}$

unforced: $x(t) = e^{At}\,x(0) \rightarrow \hat{x}(s) = (sI - A)^{-1}\,x(0)$
forced: $H(t) = Ce^{At}B \rightarrow H(s) = C(sI - A)^{-1}B$

response to inputs: $y(t) = \int_0^t H(t-\tau)\,d(\tau)\,d\tau \rightarrow \hat{y}(s) = H(s)\,\hat{d}(s)$

Unforce Response:

Non-normal $A \rightarrow \dot{x}(t) = A\,x(t)$
A doesn't commute w/ ~~operator~~ adjoint
$A A^* \neq A^* A$
$A^* \rightarrow$ complex conjegate transprose
$w_i \rightarrow$ left side eigenvector

Not in Slide show:

Adjoint: $\langle x_1, A x_2 \rangle = \langle A^{ad} \cdot x_1, x_2 \rangle$
$\langle \cdot , \cdot \rangle$ - inner producte
In $\mathbb{R}^2$: "standard" inner product
$\hookrightarrow \langle x_1, A x_2 \rangle = x_1^T \cdot A \cdot x_2 \quad (\langle x_1, x_2 \rangle = x_1^T \cdot x_2$
In $\mathbb{C}^2$: $\langle x_1, x_2 \rangle = x_1^* \cdot x_2$
$y^* = [\,\bar{y}_1, \ldots, \bar{y}_n\,]$; $\bar{y}_i = Re[y_i] - j\,Im[y_i]$
In $\mathbb{R}^2$: $\langle x_1, A x_2 \rangle = x_1^T A\, x_2$
$= (A^T x_1)^T \cdot x_2$
$= \langle A^T \cdot x_1, x_2 \rangle$

$\Rightarrow A^{ad} = A^T \quad (\text{in } \mathbb{R}^n)$
$\Rightarrow A^{ad} = A^* \quad (\text{in } \mathbb{C}^n)$

This is what a matrix does:

$$\mathbb{C}^n \xrightarrow{M} \mathbb{C}^m$$

This is what an adjunct does:

$$\mathbb{C}^n \xleftarrow{M^{ad}} \mathbb{C}^m$$

$A^{-1} = \frac{1}{\det A}\ \mathrm{ad}\cancel{\mathrm{j}}\ A$

no! adjugate! not adjoint

$\boxed{w^T \cdot A = \lambda\, w^T} \Big|^T$

Back to Slideshow...

$A = V \Lambda W^*$ (diagonalize A)

note: $e^{At} = V e^{\Lambda t} \cdot W^*$

$x(t) = e^{At}\, x(0)$

ex// $A \in \mathbb{R}^2$

$$x(t) = [v_1, v_2] \cdot \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} w_1^* \\ w_2^* \end{bmatrix} \cdot x(0) =$$

$$= [v_1, v_2] \cdot \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} w_1^* x(0) \\ w_2^* x(0) \end{bmatrix}$$

$$= [v_1, v_2] \cdot \begin{bmatrix} e^{\lambda_1 t}\, w_1^*\, x(0) \\ e^{\lambda_2 t}\, w_2^*\, x(0) \end{bmatrix}$$

$$= \sum_{i=1}^{2} e^{\lambda_i t}\, w_i^*\, x(0) \cdot v_i$$

* last time: $A = V \Lambda \underbrace{V^{-1}}_{W^*} \Rightarrow e^{At} = V e^{\Lambda t} V^{-1}$

Def. of e^{At}: $e^{At} = I + \frac{At}{1} + A^2 t^2 / 2! + \dots$

$A^2 = AA = V \Lambda \overbrace{V^{-1} \cdot V}^{I} \Lambda V^{-1} = V \Lambda^2 V^{-1}$

$A^k = V \Lambda^k V^{-1} \Rightarrow e^{At} = V \cdot \underbrace{\sum_{i=0}^{\infty} \frac{(\Lambda \cdot t)^i}{i!}}_{e^{\Lambda t}} \cdot V^{-1}$

Fact: $\alpha v = v \alpha$

$\alpha \in \mathbb{C}$ + $v \in \mathbb{C}^n$

i.p → inner product

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t}(w_i^* x(0)) v_i \xrightarrow{\text{I.P with } w_k^*} (w_k^* x(t)) = e^{\lambda_k t}(w_k^* x(0))$$

Recipe for (v_i, w_i):

$A v_i = \lambda_i v_i$

$A^T w_i = \lambda_i w_i$

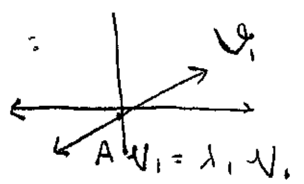
Forced Response: Preview in Slides

↳ singular value decomposition

Interpreting λ

$A v_i = \lambda_i v_i$

If $\lambda_i \in \mathbb{R}$:



note: if $\lambda_i \in \mathbb{R}$, $A \cdot v_i = \lambda_i v_i$ If $v_i \in \mathbb{C}^n$

then $A \cdot Re(v_i) = \lambda_i (Re(v_i))$

$A \cdot Im(v_i) = \lambda_i \, Im(v_i)$

If $Re(v_i) \neq 0$ or $Im(v_i) \neq 0$, we can choose them as e-vectors of A. When $\lambda_i \in \mathbb{C}$, we'll ~~have~~ need $Re(v_i)$ + $Im(v_i)$ to make ~~pys~~ physical interpretation. → (next time)