Lecture 7

Last Time:

- Solution to

  $\dot{x}(t) = A(t)\,x(t) + B(t)u(t)$

- $\Phi(t, \tau)$: state transition matrix

  $\partial\Phi/\partial t = A(t)\,\Phi(t,\tau)$ ... (1)

  $\Phi(\tau, \tau) = I$ ... (2) $\longrightarrow \begin{bmatrix} \phi_{11}(\tau,\tau) & \phi_{12}(\,) \\ \phi_{21}(\,) & \phi_{22}(\,) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Today:
- numerical computation of $\Phi(t, \tau)$
- matrix exponential
- Laplace Transform

Q: How can we solve system of matrix-valued eq (1)+(2)?
$\Phi(t, \tau) \in \mathbb{R}^{n \times n}$

ex// $\ddot{y}(t) + a_0(t)\cdot y(t) = u(t)$

choose $x_1 = y$, $x_2 = \dot{y}$ $\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0(t) & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$

$\Rightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$

mathieu equations:

$a_0(t) = K + \alpha \cdot \cos(\omega \cdot t)$

In this case, no analytical eq. for $\Phi(t, \tau)$

RHS of (1): $\begin{bmatrix} 0 & 1 \\ -a_0(t) & 0 \end{bmatrix} \cdot \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} \phi_{21} & \phi_{22} \\ -a_0(t)\phi_{11} & -a_0(t)\phi_{12} \end{bmatrix}$

LHS of (1): $\begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix}$ equate + solve!

$\Rightarrow \begin{cases} \dot{\phi}_{11} = \phi_{21} & ; \ \phi_{11}(\tau,\tau) = 1 \\ \dot{\phi}_{21} = -a_0(t)\cdot\phi_{11} & ; \ \phi_{21}(\tau,\tau) = 0 \end{cases}$

$\begin{cases} \dot{\phi}_{12} = \phi_{22} & ; \ \phi_{12}(\tau,\tau) = 0 \\ \dot{\phi}_{22} = -a_0(t)\cdot\phi_{12} & ; \ \phi_{22}(\tau,\tau) = 1 \end{cases}$

can also be written as $\ddot{\phi}_{11} + a_0(t)\,\phi_{11} = 0$

$\phi_{11}(\tau,\tau) = 1$

$\dot{\phi}_{11}(\tau,\tau) = 0$

*note: If $u \equiv 0 \Rightarrow x(t) = \phi(t, t_0) \cdot x(t_0)$

partition $\phi(t, t_0)$ into its columns:

$\phi(t, t_0) = [\, \phi_1(t, t_0) \,\vdots\, \phi_2(t, t_0) \,\vdots\, \dots \,\vdots\, \phi_n(t, t_0) \,]$

$\phi_i(t, t_0) \in \mathbb{R}^{n \times 1}$

$x(t) = \sum_{i=1}^{n} x_i(t_0) \cdot \phi_i(t, t_0)$

ex// $n = 2$

$$x(t) = [\, \phi_1 \,\vdots\, \phi_2 \,] \cdot \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

$$= \phi_1 \cdot x_1(t_0) + \phi_2 \cdot x_2(t_0)$$

*note: $x(t)$ = linear combination of columns of $\phi(t, \tau)$ w/ weights being determined by $x_i(t)$

If we set $x_j(t_0) = 1$, $x_i(t_0) = 0$ iff $i \neq j$

then $x(t) = \phi_j(t, t_0)$

"Numerical" algorithm.

$\dot{\phi}_j(t, t_0) = A(t)\, \phi_j(t, t_0)$

$\phi_j(t_0, t_0) = e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow$ jth position

(If Euler discretition: $\frac{dx}{dt} \approx \cancel{\lim_{\Delta t \to 0}}\, \frac{x(t+\Delta t) - x(t)}{\Delta t}$)

LTI case:

$\dot{x} = Ax + Bu$

A, B = constant

$\phi(t, \tau) = \phi(t - \tau)$

ex// $x(t) \in \mathbb{R}$: scalar

$\dot{x}(t) = a \cdot x(t) \quad \Rightarrow x(t) = e^{a(t - t_0)} \cdot x_0$

$x(t_0) = x_0$

Q: given $\dot{x} = A \cdot x$; $A \in \mathbb{R}^{n \times n}$ can we do something similar?

$x(t) = e^{A(t - t_0)} \cdot x(t_0)$ ... (***)

ex// $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$\Rightarrow \dot{x}_i(t) = i \cdot x_i(t) \quad ; \quad i = 1, 2$

$x_i(t) = e^{-i(t-t_0)} x_i(t_0)$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{bmatrix} \cdot \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

*note: Naive attempt of defining "matrix" exponential as exponential of individual elements of matrix fails :(

Recall: $e^{at} = \sum_{k=0}^{\infty} \frac{(a \cdot t)^k}{k!} = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \ldots$

thus $\Rightarrow e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + \frac{At}{1!} + \frac{(A^2 t^2)}{2!} + \ldots$ ⊛

In order for " " to go away from (1) we must verify that $\Phi(t, t_0) = e^{A(t-t_0)}$ where e^{At} is given by ⊛

In other words, (⊛) satisfies (1) + (2)

$e^{A \cdot 0} = I + 0 + 0 + \ldots = I$, (2) holds

LHS (2): $\partial e^{At}/\partial t = A + 2 \cdot A^2 t/2 + 3A^3 t^2/3 \cdot 2 + \ldots$

RHS (2): $= A(I + At/1! + A^2t^2/2! + \ldots) = Ae^{At} = e^{At}A$

Note: A commutes with e^{At}

$A \cdot e^{At} = e^{A \cdot t} A$

For LTI systems:

$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B \cdot u(\tau) d\tau$

in other words

$\Phi(t, t_0) = e^{A(t-t_0)}$

can set $t_0 = 0$ w/out loss of generality

Properties of e^{At}:

1.) Definition: $e^{At} = \sum_{k=0}^{\infty} (At)^k / k!$

$de^{At}/dt = A e^{at} = e^{At} A$

$e^{A \cdot 0} = I$

2.) $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

→ holds

3.) $e^{(A_1+A_2)t} = e^{A_1 t} \cdot e^{A_2 t}$

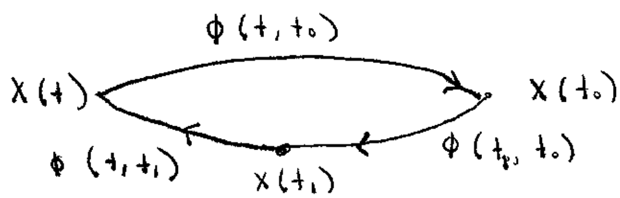
→ not true in general, but true if $A_2 A_1 = A_1 A_2$

4.) $\det \phi(t, t_0) = e^{\int_{t_0}^{t} \text{trace}(A(t))\, d\tau} \neq 0$

↳ $\text{trace} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 + 4 = \sum_{i=1}^{n} \lambda_i(A)$

$= \sum^{n} a_{ii}$

5.)



Next time: Laplace transform $\mathcal{L}\{e^{At}\} = (SI - A)^{-1}$

transfer function

$H(S) = C \cdot (SI - A)^{-1} B + D$