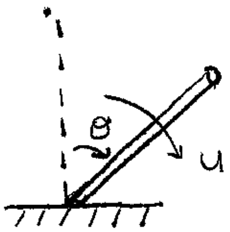
Last time: Basic system properties
- linearity
- time invariance
- causality
- memory

y = output
t = time
x = state
u = input

Today: ~~Space~~ State-Space Models
- $\dot{x} = f(x, u, t)$ ... (1) state-space eq.
- $y = g(x, u, t)$ ... (2) output eq. (static)

↳ f, g are, in general, nonlinear functions

ex// inverted pendulum



θ: angle (position)
u: torque
$\ddot{\theta} - \sin\theta = u$
choose $y = \theta$ (output = angle)
input: u (torque)
$\ddot{y} - \sin y = u$

*state → connects past behavior to future behavior of system

choose $x_1 = y \longrightarrow \dot{x}_1 = \dot{y}$
$x_2 = \dot{y} \longrightarrow \dot{x}_2 = \ddot{y} = \sin x_1 + u$

thus for inverted pendulum w/ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, functions f & g are given by $f(x,u) = \begin{bmatrix} x_2 \\ \sin x_1 + u \end{bmatrix}$
and $g(x, u) = x_1$

from (1), (2) we can determine $y(T)$ if we know:
$x(t_0)$ and $u[t_0, T]$

*state - least amount of info needed to determine output at some time (provided we know input $u[t_0, T]$)

note: if if input/output diff. eq is of form:
$F(y^{(N)}, y^{(N-1)}, \dots, \dot{y}, y, u, t) = 0$
that is, if there are no derivatives of u ($\dot{u}$, $u^{(55)}$, ...)
then we can choose "physical states".

↳ $x_1 = y$
$x_2 = \dot{y}$
$\vdots$
$x_N = y^{(N-1)}$

note: at any fixed time, $x(t)$, $u(t)$, $y(t)$ are vectors with n, m, p components.

↳ e.g. $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ + similarly for $u(t)$ + $y(t)$.

for ex// inverted pendulum, $n=2$, $m=1$, $p=1$

recall: $\dot{x} = f(x, u, t)$ ... (1)

$y = g(x, u, t)$ ... (2)

Q: What conditions a pair $(\bar{x}(t), \bar{u}(t))$ need to satisfy in order for it to solve (1)?

A: It has to satisfy (1)!

$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t), t)$

AND → $\bar{x}(t_0) = x_0$ (initial conditions)

In particular, we'll focus on nonlinear time-invariant system

$\dot{x} = f(x, u)$ ... (1)

$y = g(x, u)$ ... (2)

and steady constant trajectories (solutions) to (1).

⇒ $(\bar{x}, \bar{u})$ = constant

any constant pair $(\bar{x}, \bar{u})$ that satisfies this condition is called "equilibrium" point of (1)

What we often want to figure out is systems behavior when it's left on its own (no input).

$u \equiv 0$ → $\bar{u} = 0$

eq. points are determined by $\dot{\bar{x}} = 0 = f(\bar{x}, 0)$

for ex// inverted pendulum, $f(\bar{x}, 0) = \begin{bmatrix} \bar{x}_1 \\ \sin \bar{x}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\bar{x}_2 = 0$

$\sin \bar{x}_1 = 0 \Rightarrow \bar{x}_1 = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$

thus, eqp for inverted pend. are given by

$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$ $k = 0, \pm 1, \pm 2, \ldots$

physically, only 2 eq. points

$\bar{x}_{up} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\bar{x}_{down} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

next: want to study dynamics of fluctuations around trajectories of state-space models

represent: $x(t) = \bar{x}(t) + \tilde{x}(t)$ ... (3)

$\bar{x}$ ↑ solution of (1), $\tilde{x}$ ↑ perturbation

$u(t) = \bar{u}(t) + \tilde{u}(t)$ ... (4)

$y(t) = \bar{y}(t) + \tilde{y}(t)$ ... (5)

((3), (4), (5)) → (1), (2)

$$\dot{\bar{x}}(t) + \dot{\tilde{x}}(t) = f(\bar{x}(t) + \tilde{x}(t), u(t) + \tilde{u}(t), t)$$

$$\bar{y}(t) + \tilde{y}(t) = g(\bar{x}(t) + \tilde{x}(t), \bar{u}(t) + \tilde{u}(t), t)$$

where:

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t), t)$$

$$\bar{y}(t) = g(\bar{x}(t), \bar{u}(t), t)$$

$$\bar{x}(t_0) = x_0$$

↳ want to find equations for $\tilde{x}(t)$, $\tilde{y}(t)$

$$\dot{\tilde{x}} = f(\bar{x} + \tilde{x}, \bar{u} + \tilde{u}, t) - f(\bar{x}, \bar{u}, t) \quad \ldots \quad (\mathrm{I})$$

$$\tilde{y}(t) = g(\bar{x} + \tilde{x}, \bar{u} + \tilde{u}, t) - g(\bar{x}, \bar{u}, t) \quad \ldots \quad (\mathrm{II})$$

*NO ASSUMPTIONS (thus far)

Now: assume that $\tilde{x}$ + $\tilde{u}$ are small (in certain sense) and study dynamics of (I) + (II).

Represent f + g using Taylors Series expansion about $(\bar{x}(t), \bar{u}(t))$

$$f(\bar{x} + \tilde{x}, \bar{u} + \tilde{u}, t) = f(\bar{x}, \bar{u}, t) + \frac{\partial f}{\partial x}\Big|_{(\bar{x}(t), \bar{u}(t))} \cdot (\bar{x}(t) + \tilde{x}(t) - \bar{x}(t)) + \frac{\partial f}{\partial u}\Big|_{(\bar{x}(t), \bar{u}(t))} \cdot \tilde{u}(t) + (\text{higher order things}) \quad \ldots \quad (\mathrm{III})$$

\+ similarly for g

H.O.T - $\|\tilde{x}(t)\|^2$, $\|\tilde{u}(t)\|^2$ + higher order terms

$\|\_\_\|$ - normal of vector

$$(\mathrm{III}) \to (\mathrm{I}): \dot{\tilde{x}}(t) = \underbrace{\boxed{\frac{\partial f}{\partial x}\Big|_{(\bar{x}(t), \bar{u}(t))}}}_{A(t)} \cdot \tilde{x}(t) + \underbrace{\boxed{\frac{\partial f}{\partial u}\Big|_{(\bar{x}(t), \bar{u}(t))}}}_{B(t)} \cdot \tilde{u}(t)$$

$$\text{“(IV)”} \to (\mathrm{II}): \tilde{y}(t) = \underbrace{\boxed{\frac{\partial g}{\partial x}\Big|_{(\bar{x}(t), \bar{u}(t))}}}_{C(t)} \tilde{x}(t) + \underbrace{\boxed{\frac{\partial g}{\partial u}\Big|_{(\bar{x}(t), \bar{u}(t))}}}_{D(t)} \tilde{u}(t)$$

$$\Rightarrow \dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t) + \tilde{y}(t) = C(t)\tilde{x}(t) + D(t)\tilde{u}(t)$$