Due Monday 12/09/13; by 2pm in Xiaofan's office

1. Two pendula, coupled by a spring, are to be controlled by two equal and opposite forces u applied to the pendula bobs shown in Figure 1.

   The linearized equations of motion are given by

$$\begin{aligned} ml^2\ddot{\theta}_1 &= -ka^2(\theta_1 - \theta_2) - mgl\theta_1 - u, \\ ml^2\ddot{\theta}_2 &= -ka^2(\theta_2 - \theta_1) - mgl\theta_2 + u. \end{aligned}$$

   (a) Determine a state-space representation of the system.
   (b) Is the system controllable? Explain mathematically and physically.
   (c) Determine the range space of the controllability matrix.
   Hint: use singular value decomposition (Matlab's command svd).
   (d) Determine the transfer function from u to θ_1.
   (e) Assume $m = 1$, $l = 1$, $a = 0.5$, $k = 4$, $g = 10$. Is the system stable?

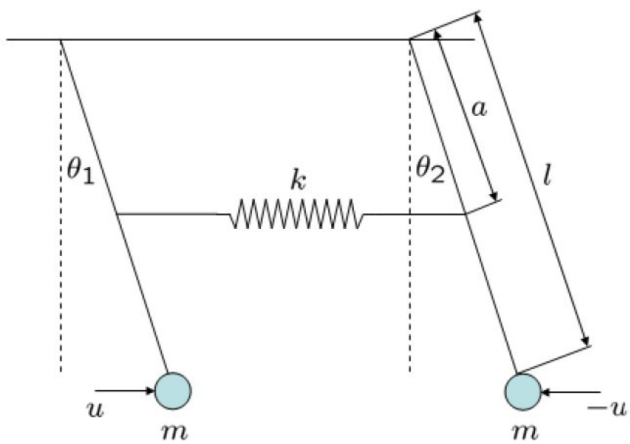

Figure 1: Two pendula coupled by a spring.

2. Consider the following pair of matrices

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

   (a) Suppose λ_1, λ_2, and λ_3 are distinct, what are the conditions on γ_1, γ_2, and γ_3 for the pair (A, B) to be controllable?
   (b) For a discrete-time system with the above given matrices A and B, $\{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3\}$, and $\gamma_1 = \gamma_2 = \gamma_3 = 1$, plot the energy (as a function of time for $k \geq 3$) of the minimum energy control necessary to reach the final state $x_f = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Comment your results.
   (c) Suppose λ_1, λ_2, and λ_3 are not necessarily distinct, how does the above condition change?
   (d) Generalize the results that you obtained for the above two cases to the pair (A, B) where

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}.$$

3. Consider the following matrices

$$A_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}.$$

(a) What are the conditions on γ_1, γ_2 and γ_3 for the pair (A_1, B_1) to be reachable?

(b) Using the matrices A_1, A_2, B_1 and B_2 we can construct the following pair of matrices

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

i. Suppose λ_1 and λ_2 are distinct, what are the conditions on γ_1, γ_2, γ_3, β_1, β_2, and β_3 for the pair (A, B) to be reachable?

ii. Suppose $\lambda_1 = \lambda_2$, what are the conditions on γ_1, γ_2, γ_3, β_1, β_2, and β_3 for the pair (A, B) to be reachable?

4. Suppose A is $n \times n$ matrix, and B is $n \times m$ matrix. Prove that (A, B) is controllable if and only if $(A + BK, B)$ is controllable for all $m \times n$ matrices K.

5. Let $\dot{x} = Ax + Bu$ be a single-input linear system with state dimension $n \geq 2$. Show that if B is an eigenvector of A then the system is not reachable.

6. Given the linear time-invariant system

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u =: Ax + Bu$$
$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} =: Cx,$$

(a) Check controllability using:
- The controllability matrix (the Kalman rank test).
- Rows of $\hat{B} := Q^{-1}B$, where Q is chosen such that $Q^{-1}AQ$ is diagonal.
- The PBH test.

(b) Identify the controllable and uncontrollable modes of the system, and convert the system to *a* Kalman controllable canonical form.