Due Tuesday 11/19/13

1. The Lienard's equation
$$\ddot{x}(t) + f(x(t))\,\dot{x}(t) + g(x(t)) = 0$$
describes a broad class of systems. Here,
   - f is an even function of $x(t)$, and $f(\cdot) \geq 0$
   - g is a monotonically increasing function and $g(0) = 0$.

   For such systems a Lyapunov function can be constructed as
$$V(x) = \int_0^{x_1} g(\tau)\,\mathrm{d}\tau + \frac{1}{2}\,x_2^2.$$
Based on this, investigate the stability of Van der Pol's equation,
$$\ddot{x}(t) + \epsilon\,(x^2(t) - 1)\,\dot{x}(t) + x(t) = 0.$$

2. Consider the following nonlinear system
$$\begin{aligned}\dot{x}_1 &= -\frac{x_2}{1+x_1^2} - 2x_1 \\ \dot{x}_2 &= \frac{x_1}{1+x_1^2}.\end{aligned}$$
   (a) Show that the origin is an equilibrium point.

   (b) Using the candidate Lyapunov function
$$V(x) = x_1^2 + x_2^2$$
   what are the stability properties of the equilibrium point?

   (c) Linearize the nonlinear system around the equilibrium point.

   (d) Obtain a suitable Lyapunov function by solving the Lyapunov equation
$$A^T P + PA = -Q,$$
   where
$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

3. For an LTI discrete-time system
$$x_{k+1} = A\,x_k$$
   (a) Use the quadratic Lyapunov function
$$V(x_k) = x_k^T P x_k$$
   with $P = P^T > 0$ to derive the conditions for stability. In other words, you should obtain an equivalent of the algebraic Lyapunov equation that we derived in class for continuous-time systems.

   (b) Using the fact that, in continuous-time, the solution to the algebraic Lyapunov equation
$$A^T P + PA = -Q$$
   is given by
$$P = \int_0^{\infty} \Phi^T(t)\,Q\,\Phi(t)\,\mathrm{d}t$$
   where $\Phi(t) = \mathrm{e}^{At}$ denotes the state-transition matrix, postulate how a solution to the algebraic Lyapunov equation in discrete-time should look like. Prove that your "guess" provides the uniques solution to the algebraic Lyapunov equation.

4. Consider the system parameterized by the scalars k and R

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -(1+k^2)/R & 0 \\ k & -(2+k^2)/R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= x_2. \end{aligned}$$

(a) For what values of k and R is this system stable?

(b) Derive the formula for the H_2 norm of this system as a function of k and R. Using this formula, plot the H_2 norm as a function of k for $R = 1$ and $R = 1000$, and as a function of R for $k = 2$.

(c) Find the solution of the unforced system (i.e. determine operator $G(t)$ that maps initial conditions to the output $y(t)$, $y(t) = G(t)x_0$).

(d) Plot the maximal singular value of $G(t)$ as a function of time (on time interval $t \in (0, 1000)$) for two different cases: a) $R = 1000$, $k = 0$; b) $R = 1000$, $k = 2$. How do these two cases compare to each other. Explain the obtained results.

5. Write a MATLAB program to compute the H_∞ norm of a SISO transfer function using the grid (in frequency) method. Test your program on the function

$$H(s) = \frac{1}{s^2 + 10^{-6}s + 1},$$

and compare your answer to the exact solution (computed by hand using the definition of the H_∞ norm).