Due Thursday 10/24/13

1. Consider the following system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 \\ \alpha A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ y &= \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{S}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the states, $y \in \mathbb{R}^{n_2}$ is the output, and α is the positive parameter.

(a) Determine conditions for stability of this system.

(b) Determine if the matrix A is normal.

(c) Find the expression for the state-transition matrix and the resolvent of the above system. The resulting state transition matrix should be partitioned conformably with the partition of the matrix A and its components should be expressed in terms of A_{11}, A_{21}, A_{22}, and α.

(d) For

$$A_{11} = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}, \quad A_{22} = -1, \quad A_{21} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

find the right and the the left eigenvectors of the matrix A. You result should be expressed in terms of the parameter α and the left eigenvectors should be normalized to satisfy bi-orthogonality condition.

(e) For the system given in part (d), determine the expression for the system's output arising from the initial conditions in x_1 and x_2. You should use the results obtained in part (d) here.

(f) Sketch the output components determined in part (e). How do your results change if α is increased? Explain your observations.

2. For the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^2 + x_1 x_2 \\ \dot{x}_2 &= -2x_2^2 + x_2 - x_1 x_2 + 2 \end{aligned}$$

(a) Show that $\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is an equilibrium point.

(b) Is $\bar{x}$ the only equilibrium point?

(c) Linearize this system around $\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and find the resolvent and the state-transition matrix of the resulting linearized system.

3. Problem 8.4 from the book, parts (a), (b) and (c) (page 78; attached). What can be said about the stability of this system?

4. Problem 9.1 from the book, parts (a), (b) (page 86; attached).

8.9 EXERCISES

8.1 (Submultiplicative matrix norms). Not all matrix norms are submultiplicative. Verify that this property does not hold for the norm

$$\|A\|_{\Delta} := \max_{1\le i\le m} \max_{1\le j\le n} |a_{ij}|,$$

which explains why this norm is not commonly used.

Hint: Consider the matrices $A = B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. □

8.2. For a given matrix A, construct vectors for which (8.2) holds for each of the three norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$. □

8.3 (Exponential of a stability matrix). Prove that when all the eigenvalues of A have strictly negative real parts, there exist constants $c, \lambda > 0$ such that

$$\|e^{At}\| \le c\, e^{-\lambda t}, \qquad \forall t \in \mathbb{R}.$$

Hint: Use the Jordan normal form. □

8.4 (Stability of LTV systems). Consider a linear system with a state-transition $\Phi(t, \tau)$ matrix for which

$$\Phi(t, 0) = \begin{bmatrix} e^t \cos 2t & e^{-2t} \sin 2t \\ -e^t \sin 2t & e^{-2t} \cos 2t \end{bmatrix}.$$

(a) Compute the state transition matrix $\Phi(t, t_0)$.

(b) Compute a matrix $A(t)$ that corresponds to the given state transition matrix.

(c) Compute the eigenvalues of $A(t)$.

(d) Classify this system in terms of Lyapunov stability.

Hint: In answering part (d), do not be misled by your answer to part (c).

8.5 (Exponential matrix transpose). Verify that $(e^{At})' = e^{A't}$.

Hint: Use the definition of matrix exponential. □

8.6 (Stability margin). Consider the continuous-time LTI system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n$$

and suppose that there exists a positive constant μ and positive-definite matrices $P, Q \in \mathbb{R}^n$ for the Lyapunov equation

$$A'P + PA + 2\mu P = -Q. \tag{8.21}$$

Show that all eigenvalues of A have real parts less than $-\mu$. A matrix A with this property is said to be *asymptotically stable with stability margin* μ.

Hint: Start by showing that all eigenvalues of A have real parts less than $-\mu$ if and only if all eigenvalues of $A + \mu I$ have real parts less than 0 (i.e., $A + \mu I$ is a stability matrix). □

Attention! BIBO stability addresses only the solutions with zero initial conditions.

Note. The factor g can be viewed as the "gain" of the system.

Definition 9.2 (BIBO stability). The system (DLTV) is said to be *(uniformly) BIBO stable* if there exists a finite constant g such that, for every input $u(\cdot)$, its forced response $y_f(\cdot)$ satisfies

$$\sup_{t\in\mathbb{N}} \|y_f(t)\| \leq g \sup_{t\in\mathbb{N}} \|u(t)\|. \qquad \square$$

Theorem 9.5 (Time domain BIBO condition). *The following two statements are equivalent.*

1. *The system* (DLTV) *is uniformly BIBO stable.*
2. *Every entry of $D(\cdot)$ is uniformly bounded and*

$$\sup_{t\geq 0} \sum_{\tau=0}^{t-1} |g_{ij}(t,\tau)| < \infty$$

for every entry $g_{ij}(t,\tau)$ of $C(t)\Phi(t,\tau)B(\tau)$. $\square$

For the following time-invariant discrete-time system

$$x^+ = Ax + Bu, \qquad y = Cx + Du, \qquad \text{(DLTI)}$$

one can conclude that the following result holds.

Theorem 9.6 (BIBO LTI conditions). *The following three statements are equivalent.*

1. *The system* (DLTI) *is uniformly BIBO stable.*
2. *For every entry $\bar{g}_{ij}(\rho)$ of $CA^\rho B$, we have*

$$\sum_{\rho=1}^{\infty} |\bar{g}_{ij}(\rho)| < \infty.$$

3. *Every pole of every entry of the transfer function of the system* (DLTI) *has magnitude strictly smaller than 1.* $\square$

9.6 EXERCISES

9.1. Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x + u.$$

(a) Compute the system's transfer function.

(b) Is the matrix A asymptotically stable, marginally stable, or unstable?

(c) Is this system BIBO stable? $\square$