1. Consider the following system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
    A_{11} & 0 \\
    \alpha A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
\]

\[y = \begin{bmatrix}
    0 & I
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix}
\]

(S)

where \(x_1 \in \mathbb{R}^{n_1}\) and \(x_2 \in \mathbb{R}^{n_2}\) are the states, \(y \in \mathbb{R}^{n_2}\) is the output, and \(\alpha\) is the positive parameter.

(a) Determine conditions for stability of this system.

(b) Determine if the matrix \(A\) is normal.

(c) Find the expression for the state-transition matrix and the resolvent of the above system. The resulting state transition matrix should be partitioned conformably with the partition of the matrix \(A\) and its components should be expressed in terms of \(A_{11}, A_{21}, A_{22}, \) and \(\alpha\).

(d) For

\[A_{11} = \begin{bmatrix}
    -1 & -2 \\
    -2 & -5
\end{bmatrix}, \quad A_{22} = -1, \quad A_{21} = \begin{bmatrix}
    1 & 2
\end{bmatrix}\]

find the right and the left eigenvectors of the matrix \(A\). You result should be expressed in terms of the parameter \(\alpha\) and the left eigenvectors should be normalized to satisfy bi-orthogonality condition.

(e) For the system given in part (d), determine the expression for the system’s output arising from the initial conditions in \(x_1\) and \(x_2\). You should use the results obtained in part (d) here.

(f) Sketch the output components determined in part (e). How do your results change if \(\alpha\) is increased? Explain your observations.

2. For the nonlinear system

\[
\begin{align*}
\dot{x}_1 &= -x_1^2 + x_1 x_2 \\
\dot{x}_2 &= -2x_2^2 + x_2 - x_1 x_2 + 2
\end{align*}
\]

(a) Show that \(\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T\) is an equilibrium point.

(b) Is \(\bar{x}\) the only equilibrium point?

(c) Linearize this system around \(\bar{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T\) and find the resolvent and the state-transition matrix of the resulting linearized system.

3. Problem 8.4 from the book, parts (a), (b) and (c) (page 78; attached). What can be said about the stability of this system?

4. Problem 9.1 from the book, parts (a), (b) (page 86; attached).
8.9 Exercises

8.1 (Submultiplicative matrix norms). Not all matrix norms are submultiplicative. Verify that this property does not hold for the norm

\[ \| A \|_\Delta := \max_{1 \leq i \leq m} \max_{1 \leq j \leq n} |a_{ij}|, \]

which explains why this norm is not commonly used.

*Hint: Consider the matrices \( A = B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \).*

8.2. For a given matrix \( A \), construct vectors for which (8.2) holds for each of the three norms \( \| \cdot \|_1 \), \( \| \cdot \|_2 \), and \( \| \cdot \|_\infty \).

8.3 (Exponential of a stability matrix). Prove that when all the eigenvalues of \( A \) have strictly negative real parts, there exist constants \( c, \lambda > 0 \) such that

\[ \| e^{At} \| \leq ce^{-\lambda t}, \quad \forall t \in \mathbb{R}. \]

*Hint: Use the Jordan normal form.*

8.4 (Stability of LTV systems). Consider a linear system with a state-transition matrix \( \Phi(t, \tau) \) for which

\[ \Phi(t, 0) = \begin{bmatrix} e^t \cos 2t & e^{-2t} \sin 2t \\ -e^t \sin 2t & e^{-2t} \cos 2t \end{bmatrix}. \]

(a) Compute the state transition matrix \( \Phi(t, t_0) \).

(b) Compute a matrix \( A(t) \) that corresponds to the given state transition matrix.

(c) Compute the eigenvalues of \( A(t) \).

(d) Classify this system in terms of Lyapunov stability.

*Hint: In answering part (d), do not be misled by your answer to part (c).*

8.5 (Exponential matrix transpose). Verify that \( (e^{At})' = e^{A't} \).

*Hint: Use the definition of matrix exponential.*

8.6 (Stability margin). Consider the continuous-time LTI system

\[ \dot{x} = Ax, \quad x \in \mathbb{R}^n \]

and suppose that there exists a positive constant \( \mu \) and positive-definite matrices \( P, Q \in \mathbb{R}^n \) for the Lyapunov equation

\[ A'P + PA + 2\mu P = -Q. \quad (8.21) \]

Show that all eigenvalues of \( A \) have real parts less than \(-\mu\). A matrix \( A \) with this property is said to be asymptotically stable with stability margin \( \mu \).

*Hint: Start by showing that all eigenvalues of \( A \) have real parts less than \(-\mu\) if and only if all eigenvalues of \( A + \mu I \) have real parts less than 0 (i.e., \( A + \mu I \) is a stability matrix).*
Definition 9.2 (BIBO stability). The system (DLTV) is said to be (uniformly) BIBO stable if there exists a finite constant $g$ such that, for every input $u(\cdot)$, its forced response $y_f(\cdot)$ satisfies
\[
\sup_{t \in \mathbb{N}} \|y_f(t)\| \leq g \sup_{t \in \mathbb{N}} \|u(t)\|.
\]

Attention! BIBO stability addresses only the solutions with zero initial conditions.

Note. The factor $g$ can be viewed as the "gain" of the system.

Theorem 9.5 (Time domain BIBO condition). The following two statements are equivalent.

1. The system (DLTV) is uniformly BIBO stable.
2. Every entry of $D(\cdot)$ is uniformly bounded and
\[
\sup_{t \geq 0} \sum_{\tau = 0}^{t-1} |g_{ij}(t, \tau)| < \infty
\]
for every entry $g_{ij}(t, \tau)$ of $C(t)\Phi(t, \tau)B(\tau)$.

For the following time-invariant discrete-time system
\[
x^+ = Ax + Bu, \quad y = Cx + Du,
\]
(DLTI)

one can conclude that the following result holds.

Theorem 9.6 (BIBO LTI conditions). The following three statements are equivalent.

1. The system (DLTI) is uniformly BIBO stable.
2. For every entry $\bar{g}_{ij}(\rho)$ of $CA^\rho B$, we have
\[
\sum_{\rho = 1}^{\infty} |\bar{g}_{ij}(\rho)| < \infty.
\]
3. Every pole of every entry of the transfer function of the system (DLTI) has magnitude strictly smaller than 1.

9.6 Exercises

9.1. Consider the system
\[
\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x + u.
\]

(a) Compute the system’s transfer function.
(b) Is the matrix $A$ asymptotically stable, marginally stable, or unstable?
(c) Is this system BIBO stable?