Reference: Zelazo & Mesbahi CDC'09, TAC'11 $\qquad$ 12-09-11

E ... incidence matrix

L ... Laplacian

$$L(K) = EKE^T = \sum_{\ell=1}^{m} k_\ell e_\ell e_\ell^T$$

Structured feedback gain $K = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_m \end{bmatrix}$

Arrive at a <u>structured optimal control problem</u>.

Let's first consider $\underbrace{\text{graphs that do not have loops}}_{\text{(trees)}}$.

Coordinate transformation:

$$\begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} E^T \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix}}_{T} x(t)$$

$\psi(t)$ .... relative difference between adjacent nodes.

$\bar{x}(t)$ .... average mode.

Then,

$$\begin{bmatrix} \dot{\psi}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} = \begin{bmatrix} -E_t^T E_t K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi(t) \\ \bar{x}(t) \end{bmatrix} + \begin{bmatrix} E_t^T \\ \frac{1}{N}\mathbb{1}^T \end{bmatrix} d(t)$$

- $\bar{x}(t)$ is preserved when $d(t) = 0$, otherwise it drifts with random walk.

$$\dot{\Psi}(t) = -E_t^T E_t K \Psi(t) + E_t^T d(t)$$

$$z(t) = \begin{bmatrix} E_t (E_t^T E_t)^{-1} \\ -E_t K \end{bmatrix} \Psi(t)$$

$\mathcal{H}_2$ - norm from d to z:

$$J(K) = \frac{1}{2} \operatorname{trace} \left(G^{-1} K^{-1} + G K \right)$$

where $G = E_t^T E_t$

$$J(K) = \frac{1}{2} \sum_{n=1}^{N-1} \left(\frac{1}{K_n g_n} + k_n g_n \right)$$

$$= \frac{1}{2} \sum_{n=1}^{N-1} \frac{1 + (k_n g_n)^2}{k_n g_n}$$

Can minimize $J(k)$ by minimizing each term $\frac{1 + (k_n g_n)^2}{k_n g_n}$, because we have seperability between the index 'n' or between nodes.

<u>So,</u> if we use incidence matrix of a tree graph, we can separate the effect of nodes on the objective function, if J is the difference between the values of each node, and then we can solve the optimal Control problem.

General undirected graphs.

incidence matrix $\quad E = [E_t \quad E_c]$

E_c: part of the incidence matrix where there is a loop (cycle).

Columns of E_c are linear combination of columns of E_t.

Equality-constrained convex optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{s.t.} \quad & Ax - b = 0 \end{aligned}$$

$$L(x,y) = f(x) + y^T(Ax - b)$$

if f is differentiable,

$$\nabla_x L(x,y) = \nabla f(x) + A^T y = 0$$

Ex $\quad f(x) = \frac{1}{2} x^T Q x \; ; \; Q = Q^T \succ 0$

$$Qx + A^T y = 0$$

$$\Leftrightarrow \boxed{x = -Q^{-1} A^T y}$$

dual ascent method

$$\begin{cases} x^{k+1} = -Q^{-1} A^T y^k \\ y^{k+1} = y^k + s^k (A x^{k+1} - b) \end{cases}$$

Advantage it may lead to distributed implementation.